高校核心课程学习指导丛书

常用积分表

CHANGYONG JIFEN BIAO

第2版

《常用积分表》编委会／编

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内 容 简 介

本书专门讲述积分方法，涵盖各种函数积分的方法，从初等函数到特殊函数，从实变函数到复变函数. 本书以方法为中心、以算例为导向，读者可在算例的引导下，逐步掌握积分的方法. 本书从易到难，由浅入深，适合不同层次、不同群体的人阅读，他们可以是初学微积分的大学生，可以是已经学过微积分的研究生，也可以是有工作经验的科学家、工程师.

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再版前言

《常用积分表》出版已十年了，承蒙读者们的眷顾，原版书已售罄，今再版，以飨读者.

此次再版，主要改正了若干已发现的错误；加大了字号和开本，更便于查阅.

若书中有错误之处，请读者指正.

《常用积分表》编委会

2019 年 2 月

前　言

这本《常用积分表》是我们在参考国内外众多数学手册和积分表的基础上，选取最基本、最常用的积分公式编纂而成的，它适合大学生使用，也可供教学和研究人员、工程技术人员参考.

本书包含最常用的初等函数和特殊函数的不定积分与定积分公式 2552 个，另外还有 203 个积分变换公式. 积分公式中遇到的所有函数（包括被积函数和积分后的函数）的定义和基本性质都可以在附录中查到.

为了节省篇幅，我们将不定积分公式中等式右边的任意常数都省略了. 例如，在积分公式

$$\int \cos x \mathrm{d}x = \sin x + C$$

中，我们省去 C，而写成

$$\int \cos x \mathrm{d}x = \sin x$$

但是，在使用这些公式做计算或练习时，一定要记住把积分常数加上去.

按照惯例，积分变量 x，t 和参数 a，b，c，d 取实数，l，m，n 取整数；当有其他限制时，相应公式后面的括号中会给出说明或注释.

本书分四个部分，分别是不定积分表、定积分表、积分变换表和附录. 不定积分表又分为初等函数的不定积分和特殊函数的不定积分两个部分，定积分表也分为初等函数的定积分和特殊函数的定积分两个部分. 在积分变换表中，我们只选入常用的拉普拉斯变换和傅里叶变换以及傅里叶正弦、余弦变换. 附录中给出了初等函数的定义及其相关公式和特殊函数的定义及其基本性质. 常用的初等函数的导数表、初等函数的级数展开表也放在附录中. 自然科学基本常数和国际单位制被编列在附录的末尾.

本书中积分公式的序号是按初等函数的不定积分、特殊函数的不定积分、初等函数的定积分、特殊函数的定积分四个部分分别编列的. 需要注释的符号和函数

都在一个小节中首次出现时给出；在同一小节中，该符号具有相同的意义，但不遍及其他小节.

因为这是一本常用的积分表，所以所有的公式都没有注明出处. 尽管如此，我们还是在书末列出了主要的参考书目，以便读者查找时参考.

我们感谢中国科学技术大学国家同步辐射实验室和中国科学技术大学出版社对出版这本工具书的大力支持.

对书中的缺点和错误，诚望读者指正.

《常用积分表》编委会

2009 年 4 月

目　录

Ⅰ 不定积分表

在所有不定积分公式中，都省略了积分常数 C. 公式中出现的变量和常量，都应在使表达式有定义的范围之内.

Ⅰ.1 初等函数的不定积分

凡在右端出现 $\ln|x|$ 或 $\ln|f(x)|$ 的积分公式中，我们都认为 x 是实变量. 当 x 是复变量时，公式中的 $\ln|x|$，$\ln|f(x)|$ 要相应地改为 $\mathrm{Ln}x$，$\mathrm{Ln}f(x)$，其中，$\mathrm{Ln}f(x)=\ln|f(x)|+\mathrm{i}\arg f(x)$，$\arg f(x)$ 是 $f(x)$ 的辐角.

Ⅰ.1.1 基本积分公式

1. $\int a\mathrm{d}x = ax$

2. $\int af(x)\mathrm{d}x = a\int f(x)\mathrm{d}x$

3. $\int \varphi[y(x)]\mathrm{d}x = \int \frac{\varphi(y)}{y'}\mathrm{d}y$ $\left(\text{这里}, y' = \frac{\mathrm{d}y}{\mathrm{d}x} \neq 0\right)$

4. $\int (u+v)\mathrm{d}x = \int u\mathrm{d}x + \int v\mathrm{d}x$ （这里，u 和 v 都是 x 的函数，以下同）

5. $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$

6. $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$

7. $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1}$ $(n \neq -1)$

8. $\int \sqrt{x^m}\mathrm{d}x = \frac{2x\sqrt{x^m}}{m+2}$ $(m \neq -2)$

9. $\int \sqrt[p]{x^m}\mathrm{d}x = \frac{px\sqrt[p]{x^m}}{m+p} \quad (m+p \neq 0)$

10. $\int \frac{\mathrm{d}x}{x} = \ln|x|$

11. $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| \left[\text{这里}, f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}, \text{以下同}\right]$

12. $\int f'(x)[f(x)]^a\mathrm{d}x = \frac{1}{a+1}[f(x)]^{a+1} \quad (a \neq -1)$

13. $\int \mathrm{e}^x\mathrm{d}x = \mathrm{e}^x$

14. $\int \mathrm{e}^{ax}\mathrm{d}x = \frac{\mathrm{e}^{ax}}{a} \quad (a \neq 0)$

15. $\int b^x\mathrm{d}x = \frac{b^x}{\ln b} \quad (b > 0,\ b \neq 1)$

16. $\int b^{ax}\mathrm{d}x = \frac{b^{ax}}{a\ln b} \quad (b > 0,\ b \neq 1,\ a \neq 0)$

17. $\int \ln x\mathrm{d}x = x\ln x - x$

18. $\int \sin x\mathrm{d}x = -\cos x$

19. $\int \cos x\mathrm{d}x = \sin x$

20. $\int \tan x\mathrm{d}x = -\ln|\cos x|$

21. $\int \cot x\mathrm{d}x = \ln|\sin x|$

22. $\int \sinh x\mathrm{d}x = \cosh x$

23. $\int \cosh x\mathrm{d}x = \sinh x$

24. $\int \tanh x\mathrm{d}x = \ln(\cosh x)$

25. $\int \coth x\mathrm{d}x = \ln|\sinh x|$

26. $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} \quad (a \neq 0)$

27. $\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{a}\operatorname{artanh}\frac{x}{a} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad (a^2 > x^2)$

28. $\int \frac{\mathrm{d}x}{x^2-a^2} = -\frac{1}{a}\operatorname{arcoth}\frac{x}{a} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| \quad (x^2 > a^2)$

29. $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} \quad (a^2 > x^2)$

30. $\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}|$

Ⅰ.1.2 包含多项式、有理分式和无理分式的不定积分

当没有特别说明时，l, m, n 为整数；$a, b, c, d, p, q, r, \alpha, \beta, \gamma$ 为实常数.

Ⅰ.1.2.1 含有 $a + bx$ 的积分

31. $\int (a + bx)^n \mathrm{d}x = \frac{(a + bx)^{n+1}}{(n+1)b} \quad (n \neq -1, 0;\ b \neq 0)$

32. $\int x(a + bx)\mathrm{d}x = \frac{x^2}{2}\left(a + \frac{2b}{3}x\right)$

33. $\int x^2(a + bx)\mathrm{d}x = \frac{x^3}{3}\left(a + \frac{3b}{4}x\right)$

34. $\int x^m(a + bx)\mathrm{d}x = \frac{x^{m+1}}{m+1}\left[a + \frac{(m+1)b}{m+2}x\right] \quad (m \neq -1, -2)$

35. $\int x(a + bx)^n \mathrm{d}x = \frac{1}{b^2(n+2)}(a + bx)^{n+2} - \frac{a}{b^2(n+1)}(a + bx)^{n+1}$

$(n \neq -1, -2)$

36. $\int x^2(a + bx)^n \mathrm{d}x = \frac{1}{b^3}\left[\frac{(a + bx)^{n+3}}{n+3} - 2a\frac{(a + bx)^{n+2}}{n+2} + a^2\frac{(a + bx)^{n+1}}{n+1}\right]$

$(n \neq -1, -2, -3)$

37. $\int x^m(a + bx)^n \mathrm{d}x = \frac{x^{m+1}(a + bx)^n}{m + n + 1} + \frac{an}{m + n + 1}\int x^m(a + bx)^{n-1}\mathrm{d}x$

$= \frac{1}{a(n+1)}\left[-x^{m+1}(a + bx)^{n+1} + (m + n + 2)\int x^m(a + bx)^{n+1}\mathrm{d}x\right]$

$= \frac{1}{b(m + n + 1)}\left[x^m(a + bx)^{n+1} - ma\int x^{m-1}(a + bx)^n\mathrm{d}x\right]$

38. $\int \frac{a + bx}{x}\mathrm{d}x = a\ln|x| + bx$

39. $\int \frac{a + bx}{x^2}\mathrm{d}x = b\ln|x| - \frac{a}{x}$

40. $\int \frac{a + bx}{x^n}\mathrm{d}x = -\frac{a}{(n-1)x^{n-1}} - \frac{b}{(n-2)x^{n-2}} \quad (n > 2)$

41. $\int \frac{\mathrm{d}x}{a + bx} = \frac{1}{b}\ln|a + bx|$

42. $\int \frac{\mathrm{d}x}{(a + bx)^2} = -\frac{1}{b(a + bx)}$

43. $\int \frac{\mathrm{d}x}{(a+bx)^n} = -\frac{1}{(n-1)b(a+bx)^{n-1}} \quad (n \neq 0,1)$

44. $\int \frac{x}{a+bx}\mathrm{d}x = \frac{x}{b} - \frac{a}{b^2}\ln|a+bx|$

45. $\int \frac{x}{(a+bx)^2}\mathrm{d}x = \frac{1}{b^2}\left(\ln|a+bx| + \frac{a}{a+bx}\right)$

46. $\int \frac{x}{(a+bx)^n}\mathrm{d}x = \frac{1}{b^2}\left[-\frac{1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}}\right]$

$(n \neq 1,2)$

47. $\int \frac{x^2}{a+bx}\mathrm{d}x = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\right]$

48. $\int \frac{x^2}{(a+bx)^2}\mathrm{d}x = \frac{1}{b^3}\left(a+bx-2a\ln|a+bx| - \frac{a^2}{a+bx}\right)$

49. $\int \frac{x^2}{(a+bx)^3}\mathrm{d}x = \frac{1}{b^3}\left[\ln|a+bx| + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]$

50. $\int \frac{x^2}{(a+bx)^n}\mathrm{d}x$

$$= \frac{1}{b^3}\left[-\frac{1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right]$$

$(n \neq 1,2,3)$

51. $\int \frac{x^m}{a+bx}\mathrm{d}x = \frac{1}{b}\left[\left(-\frac{a}{b}\right)^m\ln|a+bx| + x^m\sum_{k=0}^{m-1}\frac{1}{m-k}\left(-\frac{a}{bx}\right)^k\right]$

52. $\int \frac{x^m}{(a+bx)^2}\mathrm{d}x = \sum_{k=1}^{m-1}(-1)^{k-1}\frac{ka^{k-1}x^{m-k}}{(m-k)b^{k+1}} + (-1)^{m-1}\frac{a^m}{b^{m+1}(a+bx)}$

$$+ (-1)^{m+1}\frac{ma^{m-1}}{b^{m+1}}\ln|a+bx|$$

53. $\int \frac{x^m}{(a+bx)^n}\mathrm{d}x = \frac{1}{b^{m+1}}\sum_{k=0}^{m}\binom{m}{k}\frac{(-a)^k(a+bx)^{m-n-k+1}}{m-n-k+1}$

$\left[\text{这里}, m-n-k+1=0\text{ 的项要替换成}\binom{m}{n-1}(-a)^{m-n+1}\ln|a+bx|\right]$

54. $\int \frac{\mathrm{d}x}{x(a+bx)} = -\frac{1}{a}\ln\left|\frac{a+bx}{x}\right|$

55. $\int \frac{\mathrm{d}x}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right|$

56. $\int \frac{\mathrm{d}x}{x(a+bx)^n} = -\frac{1}{a^n}\ln\left|\frac{a+bx}{x}\right| + \frac{1}{a^n}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-bx)^k}{k(a+bx)^k} \quad (n \neq 0)$

57. $\int \frac{\mathrm{d}x}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right|$

58. $\int \frac{\mathrm{d}x}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\ln\left|\frac{a+bx}{x}\right|$

59. $\int \frac{\mathrm{d}x}{x^2(a+bx)^n}$

$$= \frac{1}{a^{n+1}}\left[n b\ln\left|\frac{a+bx}{x}\right| - \frac{a+bx}{x} + \frac{a+bx}{x}\sum_{k=2}^{n}\binom{n}{k}\frac{(-bx)^k}{(k-1)(a+bx)^k}\right]$$

$(n \neq 0,1)$

60. $\int \frac{\mathrm{d}x}{x^m(a+bx)} = \frac{1}{b}\left[\left(-\frac{b}{a}\right)^m \ln|a+bx| - \frac{1}{x^m}\sum_{k=1}^{m-1}\frac{1}{m+1}\left(-\frac{a}{bx}\right)^k\right]$

61. $\int \frac{\mathrm{d}x}{x^m(a+bx)^n} = -\frac{1}{a^{m+n-1}}\sum_{k=0}^{m+n-2}\binom{m+n-2}{k}\frac{(a+bx)^{m-k-1}(-b)^k}{(m-k-1)^{m-k-1}x^{m-k-1}}$

$\left[\text{这里}, m-k-1=0\text{ 的项要替换成}\binom{m+n-2}{n-1}(-b)^{m-1}\ln\left|\frac{a+bx}{x}\right|\right]$

Ⅰ.1.2.2 含有 $a+bx$ 和 $c+dx$ 的积分

令 $u=a+bx$，$v=c+dx$ 和 $k=ad-bc$，$k\neq 0$.（如果 $k=0$，则 $v=\frac{c}{a}u$，这是 Ⅰ.1.2.1 小节的情形，应该使用其中相应的公式.）

62. $\int \frac{\mathrm{d}x}{uv} = \frac{1}{k}\ln\left|\frac{v}{u}\right|$

63. $\int \frac{x}{uv}\mathrm{d}x = \frac{1}{k}\left(\frac{a}{b}\ln|u| - \frac{c}{d}\ln|v|\right)$

64. $\int \frac{x^2}{uv}\mathrm{d}x = \frac{x}{bd} - \frac{a}{b^2d}\ln|u| - \frac{c}{bd^2}\ln|v| + \frac{ac}{kbd}\ln\left|\frac{v}{u}\right|$

65. $\int \frac{\mathrm{d}x}{u^2v} = \frac{1}{k}\left(\frac{1}{u} + \frac{d}{k}\ln\left|\frac{v}{u}\right|\right)$

66. $\int \frac{\mathrm{d}x}{u^2v^2} = -\frac{1}{k^2}\left(\frac{b}{u} + \frac{d}{v}\right) - \frac{2bd}{k^3}\ln\left|\frac{v}{u}\right|$

67. $\int \frac{x}{u^2v}\mathrm{d}x = -\frac{a}{bku} - \frac{c}{k^2}\ln\left|\frac{v}{u}\right|$

68. $\int \frac{x}{u^2v^2}\mathrm{d}x = \frac{1}{k^2}\left(\frac{a}{u} + \frac{c}{v}\right) + \frac{ad+bc}{k^3}\ln\left|\frac{v}{u}\right|$

69. $\int \frac{x^2}{u^2v}\mathrm{d}x = \frac{a^2}{b^2ku} + \frac{1}{k^2}\left[\frac{c^2}{d}\ln|v| + \frac{a(k-bc)}{b^2}\ln|u|\right]$

70. $\int \frac{x^2}{u^2v^2}\mathrm{d}x = -\frac{1}{k^2}\left(\frac{a^2}{bu} + \frac{c^2}{dv}\right) - \frac{2ac}{k^3}\ln\left|\frac{v}{u}\right|$

71. $\int \frac{\mathrm{d}x}{u^nv^m} = \frac{1}{k(m-1)}\left[-\frac{1}{u^{n-1}v^{m-1}} - b(m+n-2)\int\frac{\mathrm{d}x}{u^nv^{m-1}}\right]$

72. $\int u^mv^n\mathrm{d}x = \frac{u^{m+1}v^n}{(m+n+1)b} + \frac{nk}{(n+n+1)b}\int u^mv^{n-1}\mathrm{d}x$

73. $\int \frac{u}{v}\mathrm{d}x = \frac{bx}{d} + \frac{k}{d^2}\ln|v|$

74. $\int \frac{u^m}{v}\mathrm{d}x = \sum_{r=0}^{m-1} \frac{k^r u^{m-r}}{(m-r)d^{r+1}} + \frac{k^m}{d^{m+1}}\ln|v|$

75. $$\begin{aligned}\int \frac{u^m}{v^n}\mathrm{d}x &= -\frac{1}{k(n-1)}\left[\frac{u^{m+1}}{v^{n-1}} + b(n-m-2)\int \frac{u^m}{v^{n-1}}\mathrm{d}x\right] \\ &= -\frac{1}{d(n-m-1)}\left(\frac{u^m}{v^{n-1}} + mk\int \frac{u^{m-1}}{v^n}\mathrm{d}x\right) \\ &= -\frac{1}{d(n-1)}\left(\frac{u^m}{v^{n-1}} - mb\int \frac{u^{m-1}}{v^{n-1}}\mathrm{d}x\right) \quad (n \neq 1)\end{aligned}$$

Ⅰ.1.2.3 含有 $a + bx^n$ 的积分

76. $$\int \frac{\mathrm{d}x}{a+bx^2} = \begin{cases} \frac{1}{\sqrt{ab}}\arctan\frac{x\sqrt{ab}}{a} & (ab > 0) \\ \frac{1}{2\sqrt{-ab}}\ln\left|\frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}\right| & (ab < 0) \\ \frac{1}{\sqrt{-ab}}\operatorname{artanh}\frac{x\sqrt{-ab}}{a} & (ab < 0) \end{cases}$$

77. $\int \frac{\mathrm{d}x}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{\mathrm{d}x}{a+bx^2}$

78. $$\begin{aligned}\int \frac{\mathrm{d}x}{(a+bx^2)^{m+1}} &= \frac{1}{2ma}\frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma}\int \frac{\mathrm{d}x}{(a+bx^2)^m} \\ &= \frac{(2m)!}{(m!)^2}\left[\frac{x}{2a}\sum_{r=1}^{m}\frac{r!(r-1)!}{(4a)^{m-r}(2r)!(a+bx^2)^r} + \frac{1}{(4a)^m}\int \frac{\mathrm{d}x}{a+bx^2}\right]\end{aligned}$$

79. $\int \frac{x}{a+bx^2}\mathrm{d}x = \frac{1}{2b}\ln|a+bx^2|$

80. $\int \frac{x}{(a+bx^2)^{m+1}}\mathrm{d}x = -\frac{1}{2mb(a+bx^2)^m}$

81. $\int \frac{x^2}{a+bx^2}\mathrm{d}x = \frac{x}{b} - \frac{a}{b}\int \frac{\mathrm{d}x}{a+bx^2}$

82. $\int \frac{x^2}{(a+bx^2)^{m+1}}\mathrm{d}x = -\frac{x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{\mathrm{d}x}{(a+bx^2)^m}$

83. $\int \frac{\mathrm{d}x}{x(a+bx^2)} = \frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right|$

84. $$\begin{aligned}\int \frac{\mathrm{d}x}{x(a+bx^2)^{m+1}} &= \frac{1}{2ma(a+bx^2)^m} + \frac{1}{a}\int \frac{\mathrm{d}x}{x(a+bx^2)^m} \\ &= \frac{1}{2a^{m+1}}\left[\sum_{r=1}^{m}\frac{a^r}{r(a+bx^2)^r} + \ln\left|\frac{x^2}{a+bx^2}\right|\right]\end{aligned}$$

85. $\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$

86. $\int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}$

87. $\int \frac{dx}{a+bx^3} = \frac{k}{3a}\left[\frac{1}{2}\ln\left|\frac{(k+x)^3}{a+bx^3}\right| + \sqrt{3}\arctan\frac{2x-k}{k\sqrt{3}}\right]\left(这里, k = \sqrt[3]{\frac{a}{b}}\right)$

88. $\int \frac{x}{a+bx^3}dx = \frac{1}{3bk}\left[\frac{1}{2}\ln\left|\frac{a+bx^3}{(k+x)^3}\right| + \sqrt{3}\arctan\frac{2x-k}{k\sqrt{3}}\right]\left(这里, k = \sqrt[3]{\frac{a}{b}}\right)$

89. $\int \frac{x^2}{a+bx^3}dx = \frac{1}{3b}\ln|a+bx^3|$

90. $\int \frac{dx}{(a+bx^3)^2} = \frac{x}{3a(a+bx^3)} + \frac{2}{3a}\int \frac{dx}{a+bx^3}$

91. $\int \frac{x}{(a+bx^3)^2}dx = \frac{x^2}{3a(a+bx^3)} + \frac{1}{3a}\int \frac{x}{a+bx^3}dx$

92. $\int \frac{x^2}{(a+bx^3)^2}dx = -\frac{1}{3b(a+bx^3)}$

93. $\int \frac{dx}{x(a+bx^3)} = \frac{1}{3a}\ln\left|\frac{x^3}{a+bx^3}\right|$

94. $\int \frac{dx}{x^2(a+bx^3)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x}{a+bx^3}dx$

95. $\int \frac{dx}{x(a+bx^3)^2} = \frac{1}{3a(a+bx^3)} + \frac{1}{3a^2}\ln\left|\frac{x^3}{a+bx^3}\right|$

96. $\int \frac{dx}{x^2(a+bx^3)^2} = -\left(\frac{1}{ax} + \frac{4bx^2}{3a^2}\right)\frac{1}{a+bx^3} - \frac{4b}{3a^2}\int \frac{x}{a+bx^3}dx$

97. $\int \frac{dx}{a+bx^4}$

$$= \begin{cases} \frac{k}{2a}\left(\frac{1}{2}\ln\frac{x^2+2kx+2k^2}{x^2-2kx+2k^2} + \arctan\frac{2kx}{2k^2-x^2}\right) & \left(ab>0,\ k=\sqrt[4]{\frac{a}{4b}}\right) \\ \frac{k}{2a}\left(\frac{1}{2}\ln\left|\frac{x+k}{x-k}\right| + \arctan\frac{x}{k}\right) & \left(ab<0,\ k=\sqrt[4]{-\frac{a}{b}}\right) \end{cases}$$

98. $$\int \frac{x}{a+bx^4}dx = \begin{cases} \frac{1}{2bk}\arctan\frac{x}{k} & \left(ab>0,\ k=\sqrt{\frac{a}{b}}\right) \\ \frac{1}{4bk}\ln\left|\frac{x^2-k}{x^2+k}\right| & \left(ab<0,\ k=\sqrt{-\frac{a}{b}}\right) \end{cases}$$

99. $\int \frac{x^2}{a+bx^4}dx$

$$= \begin{cases} \frac{1}{4bk}\left(\frac{1}{2}\ln\frac{x^2-2kx+2k^2}{x^2+2kx+2k^2} + \arctan\frac{2kx}{2k^2-x^2}\right) & \left(ab>0,\ k=\sqrt[4]{\frac{a}{4b}}\right) \\ \frac{1}{4bk}\left(\ln\left|\frac{x-k}{x+k}\right| + 2\arctan\frac{x}{k}\right) & \left(ab<0,\ k=\sqrt[4]{-\frac{a}{b}}\right) \end{cases}$$

100. $\int \frac{x^3}{a+bx^4}\mathrm{d}x = \frac{1}{4b}\ln|a+bx^4|$

101. $\int \frac{\mathrm{d}x}{x(a+bx^n)} = \frac{1}{na}\ln\left|\frac{x^n}{a+bx^n}\right|$

102. $\int \frac{\mathrm{d}x}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{\mathrm{d}x}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^n}{(a+bx^n)^{m+1}}\mathrm{d}x$

103. $\int \frac{x^m}{(a+bx^n)^{p+1}}\mathrm{d}x = \frac{1}{b}\int \frac{x^{m-n}}{(a+bx^n)^p}\mathrm{d}x - \frac{a}{b}\int \frac{x^{m-n}}{(a+bx^n)^{p+1}}\mathrm{d}x$

104. $\int \frac{\mathrm{d}x}{x^m(a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{\mathrm{d}x}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{\mathrm{d}x}{x^{m-n}(a+bx^n)^{p+1}}$

105. $\int x^m(a+bx^n)^p\mathrm{d}x$

$$= \frac{1}{b(np+m+1)}\left[x^{m-n+1}(a+bx^n)^{p+1} - a(m-n+1)\int x^{m-n}(a+bx^n)^p\mathrm{d}x\right]$$

$$= \frac{1}{np+m+1}\left[x^{m+1}(a+bx^n)^p + n\,ap\int x^m(a+bx^n)^{p-1}\mathrm{d}x\right]$$

$$= \frac{1}{a(m+1)}\left[x^{m+1}(a+bx^n)^{p+1} - b(m+1+np+n)\int x^{m+n}(a+bx^n)^p\mathrm{d}x\right]$$

$$= \frac{1}{n\,a(p+1)}\left[-x^{m+1}(a+bx^n)^{p+1} + (m+1+np+n)\int x^m(a+bx^n)^{p+1}\mathrm{d}x\right]$$

Ⅰ.1.2.4 含有 $1 \pm x^n$ 的积分

106. $\int \frac{\mathrm{d}x}{1+x} = \ln|1+x|$

107. $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x$

108. $\int \frac{\mathrm{d}x}{1+x^3} = \frac{1}{3}\ln\frac{|1+x|}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2-x}$

109. $\int \frac{\mathrm{d}x}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{1-x^2}$

110. $\int \frac{\mathrm{d}x}{1+x^n} = -\frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1}P_k\cos\frac{(2k+1)\pi}{n} + \frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1}Q_k\sin\frac{(2k+1)\pi}{n}$（$n$ 为正偶数）

$$\left\{\text{这里}, P_k = \frac{1}{2}\ln\left[x^2 - 2x\cos\frac{(2k+1)\pi}{n} + 1\right], Q_k = \arctan\frac{x-\cos\frac{(2k+1)\pi}{n}}{\sin\frac{(2k+1)\pi}{n}}\right\}$$

111. $\int \frac{\mathrm{d}x}{1+x^n} = \frac{1}{n}\ln|1+x| - \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}}P_k\cos\frac{(2k+1)\pi}{n}$

$$+\frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}}Q_k\sin\frac{(2k+1)\pi}{n}\quad(n\text{ 为正奇数})$$

$$\left\{\text{这里},P_k=\frac{1}{2}\ln\left[x^2-2x\cos\frac{(2k+1)\pi}{n}+1\right],Q_k=\arctan\frac{x-\cos\frac{(2k+1)\pi}{n}}{\sin\frac{(2k+1)\pi}{n}}\right\}$$

112. $\int\frac{\mathrm{d}x}{1-x}=-\ln|1-x|$

113. $\int\frac{\mathrm{d}x}{1-x^2}=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|=\operatorname{artanh}x\quad(|x|<1)$

114. $\int\frac{\mathrm{d}x}{1-x^3}=\frac{1}{3}\ln\frac{\sqrt{1+x+x^2}}{|1-x|}+\frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2+x}$

115. $\int\frac{\mathrm{d}x}{1-x^4}=\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{2}\arctan x$

116. $\int\frac{\mathrm{d}x}{1-x^n}=\frac{1}{n}\ln\left|\frac{1+x}{1-x}\right|-\frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1}P_k\cos\frac{2k\pi}{n}+\frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1}Q_k\sin\frac{2k\pi}{n}$（$n$ 为正偶数）

$$\left[\text{这里},P_k=\frac{1}{2}\ln\left(x^2-2x\cos\frac{2k\pi}{n}+1\right),Q_k=\arctan\frac{x-\cos\frac{2k\pi}{n}}{\sin\frac{2k\pi}{n}}\right]$$

117. $\int\frac{\mathrm{d}x}{1-x^n}=-\frac{1}{n}\ln|1-x|+\frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}}P_k\cos\frac{(2k+1)\pi}{n}+\frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}}Q_k\sin\frac{(2k+1)\pi}{n}$

（n 为正奇数）

$$\left\{\text{这里},P_k=\frac{1}{2}\ln\left[x^2+2x\cos\frac{(2k+1)\pi}{n}+1\right],Q_k=\arctan\frac{x+\cos\frac{(2k+1)\pi}{n}}{\sin\frac{(2k+1)\pi}{n}}\right\}$$

118. $\int\frac{x}{1+x}\mathrm{d}x=x-\ln|1+x|$

119. $\int\frac{x}{1+x^2}\mathrm{d}x=\frac{1}{2}\ln(1+x^2)$

120. $\int\frac{x}{1+x^3}\mathrm{d}x=-\frac{1}{6}\ln\frac{(1+x)^2}{1-x+x^2}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$

121. $\int\frac{x}{1+x^4}\mathrm{d}x=\frac{1}{2}\arctan x^2$

122. $\int\frac{x^{m-1}}{1+x^{2n}}\mathrm{d}x=-\frac{1}{2n}\sum_{k=1}^{n}\cos\frac{m(2k-1)\pi}{2n}\ln\left[1-2x\cos\frac{(2k-1)\pi}{2n}+x^2\right]$

$$+\frac{1}{n}\sum_{k=1}^{n}\sin\frac{m(2k-1)\pi}{2n}\arctan\frac{x-\cos\frac{(2k-1)\pi}{2n}}{\sin\frac{(2k-1)\pi}{2n}}$$

(m 和 n 皆为自然数,且 $m \leqslant 2n$)

123. $$\int \frac{x^{m-1}}{1+x^{2n+1}}\mathrm{d}x = \frac{(-1)^{m+1}}{2n+1}\ln|1+x| - \frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m(2k-1)\pi}{2n+1}\ln\left[1-2x\cos\frac{(2k-1)\pi}{2n+1}+x^2\right] + \frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m(2k-1)\pi}{2n+1}\arctan\frac{x-\cos\frac{(2k-1)\pi}{2n+1}}{\sin\frac{(2k-1)\pi}{2n+1}}$$

(m 和 n 皆为自然数,且 $m \leqslant 2n$)

124. $$\int \frac{x}{1-x}\mathrm{d}x = -x-\ln|1-x|$$

125. $$\int \frac{x}{1-x^2}\mathrm{d}x = -\frac{1}{2}\ln|1-x^2|$$

126. $$\int \frac{x}{1-x^3}\mathrm{d}x = -\frac{1}{6}\ln\frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

127. $$\int \frac{x}{1-x^4}\mathrm{d}x = \frac{1}{4}\ln\left|\frac{1+x^2}{1-x^2}\right|$$

128. $$\int \frac{x^{m-1}}{1-x^{2n}}\mathrm{d}x = \frac{1}{2n}\left[(-1)^{m+1}\ln|1+x| - \ln|1-x|\right] - \frac{1}{2n}\sum_{k=1}^{n-1}\cos\frac{km\pi}{n}\ln\left(1-2x\cos\frac{k\pi}{n}+x^2\right) + \frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{km\pi}{n}\arctan\frac{x-\cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}$$

(m 和 n 皆为自然数,且 $m < 2n$)

129. $$\int \frac{x^{m-1}}{1-x^{2n+1}}\mathrm{d}x = -\frac{1}{2n+1}\ln|1-x| + \frac{(-1)^{m+1}}{2n+1}\sum_{k=1}^{n}\cos\frac{m(2k-1)\pi}{2n+1}\ln\left[1+2x\cos\frac{(2k-1)\pi}{2n}+x^2\right] + \frac{(-1)^{m+1}\cdot 2}{2n+1}\sum_{k=1}^{n}\sin\frac{m(2k-1)\pi}{2n+1}\arctan\frac{x+\cos\frac{(2k-1)\pi}{2n+1}}{\sin\frac{(2k-1)\pi}{2n+1}}$$

(m 和 n 皆为自然数,且 $m \leqslant 2n$)

Ⅰ.1.2.5 含有 c^2+x^2 的积分

130. $$\int \frac{\mathrm{d}x}{c^2+x^2} = \frac{1}{c}\arctan\frac{x}{c}$$

131. $\int \frac{dx}{(c^2+x^2)^2} = \frac{1}{2c^3}\left(\frac{cx}{c^2+x^2} + \arctan\frac{x}{c}\right)$

132. $\int \frac{dx}{(c^2+x^2)^n} = \frac{x}{2(n-1)c^2(c^2+x^2)^{n-1}} + \frac{2n-3}{2(n-1)c^2}\int \frac{dx}{(c^2+x^2)^{n-1}}$
$(n \neq 1)$

133. $\int \frac{x}{c^2+x^2}dx = \frac{1}{2}\ln(c^2+x^2)$

134. $\int \frac{x}{(c^2+x^2)^2}dx = -\frac{1}{2(c^2+x^2)}$

135. $\int \frac{x}{(c^2+x^2)^{n+1}}dx = -\frac{1}{2n(c^2+x^2)^n}$

136. $\int \frac{x^2}{c^2+x^2}dx = x - c\arctan\frac{x}{c}$

137. $\int \frac{x^2}{(c^2+x^2)^2}dx = -\frac{x}{2(c^2+x^2)} + \frac{1}{2c}\arctan\frac{x}{c}$

138. $\int \frac{x^2}{(c^2+x^2)^n}dx = -\frac{x}{2(n-1)(c^2+x^2)^{n-1}} + \frac{1}{2(n-1)}\int \frac{dx}{(c^2+x^2)^{n-1}}$
$(n \neq 1)$

139. $\int \frac{x^3}{c^2+x^2}dx = \frac{x^2}{2} - \frac{c^2}{2}\ln(c^2+x^2)$

140. $\int \frac{x^3}{(c^2+x^2)^2}dx = \frac{c^2}{2(c^2+x^2)} + \frac{1}{2}\ln(c^2+x^2)$

141. $\int \frac{x^3}{(c^2+x^2)^n}dx = -\frac{1}{2(n-2)(c^2+x^2)^{n-2}} + \frac{c^2}{2(n-1)(c^2+x^2)^{n-1}}$
$(n \neq 1,2)$

142. $\int \frac{x^m}{(c^2+x^2)^n}dx = -\frac{x^{m-1}}{2(n-1)(c^2+x^2)^{n-1}} + \frac{m-1}{2(n-1)}\int \frac{x^{m-2}}{(c^2+x^2)^{n-1}}dx$
$(n \neq 1)$

143. $\int \frac{dx}{x(c^2+x^2)} = \frac{1}{2c^2}\ln\frac{x^2}{c^2+x^2}$

144. $\int \frac{dx}{x(c^2+x^2)^2} = \frac{1}{2c^2(c^2+x^2)} + \frac{1}{2c^4}\ln\frac{x^2}{c^2+x^2}$

145. $\int \frac{dx}{x(c^2+x^2)^n} = \frac{1}{2(n-1)c^2(c^2+x^2)^{n-1}} + \frac{1}{c^2}\int \frac{dx}{x(c^2+x^2)^{n-1}}$ $(n \neq 1)$

146. $\int \frac{dx}{x^2(c^2+x^2)} = -\frac{1}{c^2x} - \frac{1}{c^3}\arctan\frac{x}{c}$

147. $\int \frac{dx}{x^2(c^2+x^2)^2} = -\frac{1}{c^4x} - \frac{x}{2c^4(c^2+x^2)} - \frac{3}{2c^5}\arctan\frac{x}{c}$

148. $\int \frac{dx}{x^2(c^2+x^2)^n} = -\frac{1}{c^2x(c^2+x^2)^{n-1}} - \frac{2n-1}{c^2}\int \frac{dx}{(c^2+x^2)^n}$

149. $\int \frac{dx}{x^m(c^2+x^2)^n} = -\frac{1}{(m-1)c^2x^{m-1}(c^2+x^2)^{n-1}} - \frac{m+2n-3}{(m-1)c^2}\int \frac{dx}{x^{m-2}(c^2+x^2)^n}$ $(m \neq 1)$

150. $\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\arctan\frac{bx}{a}$

151. $\int \frac{e+fx}{a^2+x^2}dx = f\ln\sqrt{a^2+x^2} + \frac{e}{a}\arctan\frac{x}{a}$

152. $\int \frac{dx}{(e+fx)(a^2+x^2)} = \frac{1}{e^2+a^2f^2}\left[f\ln|e+fx| - \frac{f}{2}\ln(a^2+x^2) + \frac{e}{a}\arctan\frac{x}{a}\right]$ $(e^2 \neq -a^2f^2)$

Ⅰ.1.2.6 含有 c^2-x^2 的积分

153. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\ln\left|\frac{c+x}{c-x}\right|$ $(c^2 > x^2)$

154. $\int \frac{dx}{(c^2-x^2)^2} = \frac{1}{2c^3}\left(\frac{cx}{c^2-x^2} + \ln\sqrt{\frac{c+x}{c-x}}\right)$

155. $\int \frac{dx}{(c^2-x^2)^n} = \frac{1}{2c^2(n-1)}\left[\frac{x}{(c^2-x^2)^{n-1}} + (2n-3)\int \frac{dx}{(c^2-x^2)^{n-1}}\right]$ $(n \neq 1)$

156. $\int \frac{x}{c^2-x^2}dx = -\frac{1}{2}\ln|c^2-x^2|$

157. $\int \frac{x}{(c^2-x^2)^2}dx = \frac{1}{2(c^2-x^2)}$

158. $\int \frac{x}{(c^2-x^2)^{n+1}}dx = \frac{1}{2n(c^2-x^2)^n}$

159. $\int \frac{x^2}{c^2-x^2}dx = -x + \frac{c}{2}\ln\left|\frac{c+x}{c-x}\right|$

160. $\int \frac{x^2}{(c^2-x^2)^2}dx = \frac{x}{2(c^2-x^2)} - \frac{1}{4c}\ln\left|\frac{c+x}{c-x}\right|$

161. $\int \frac{x^2}{(c^2-x^2)^n}dx = \frac{x}{2(n-1)(c^2-x^2)^{n-1}} - \frac{1}{2(n-1)}\int \frac{dx}{(c^2-x^2)^{n-1}}$ $(n \neq 1)$

162. $\int \frac{x^m}{(c^2-x^2)^n}dx = \frac{x^{m-1}}{2(n-1)(c^2-x^2)^{n-1}} - \frac{m-1}{2(n-1)}\int \frac{x^{m-2}}{(c^2-x^2)^{n-1}}dx$ $(n \neq 1)$

163. $\int \frac{dx}{x(c^2-x^2)} = \frac{1}{2c^2}\ln\left|\frac{x^2}{c^2-x^2}\right|$

164. $\int \frac{dx}{x(c^2-x^2)^2} = \frac{1}{2c^2(c^2-x^2)} + \frac{1}{2c^4}\ln\left|\frac{x^2}{c^2-x^2}\right|$

165. $\int \frac{dx}{x(c^2-x^2)^n} = \frac{1}{2(n-1)c^2(c^2-x^2)^{n-1}} + \frac{1}{c^2}\int \frac{dx}{x(c^2-x^2)^{n-1}}$ $(n \neq 1)$

166. $\int \frac{dx}{x^2(c^2-x^2)} = -\frac{1}{c^2x} + \frac{1}{c^3}\text{artanh}\,\frac{x}{c}$

167. $\int \frac{dx}{x^2(c^2-x^2)^2} = -\frac{1}{c^4x} + \frac{1}{2c^4(c^2-x^2)} + \frac{3}{2c^5}\text{artanh}\,\frac{x}{c}$

168. $\int \frac{dx}{x^m(c^2-x^2)^n} = -\frac{1}{(m-1)c^2x^{m-1}} + \frac{m+2n-3}{(m-1)c^2}\int \frac{dx}{x^{m-2}(c^2-x^2)^n} \quad (m \neq 1)$

169. $\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right|$

170. $\int \frac{e+fx}{a^2-x^2}dx = -f\ln\sqrt{a^2-x^2} + \frac{e}{a}\ln\sqrt{\frac{a+x}{a-x}}$

171. $\int \frac{dx}{(e+fx)(a^2-x^2)}$

$$= \frac{1}{a^2f^2-e^2}\left(f\ln|e+fx| - \frac{f}{2}\ln|a^2-x^2| - \frac{e}{a}\text{artanh}\,\frac{x}{a}\right)$$

$(e^2 \neq a^2f^2)$

Ⅰ.1.2.7 含有 $c^3 \pm x^3$ 的积分

172. $\int \frac{dx}{c^3 \pm x^3} = \pm\frac{1}{6c^2}\ln\left|\frac{(c\pm x)^3}{c^3\pm x^3}\right| + \frac{1}{c^3\sqrt{3}}\arctan\frac{2x\mp c}{c\sqrt{3}}$

173. $\int \frac{dx}{(c^3\pm x^3)^2} = \frac{x}{3c^3(c^3\pm x^3)} + \frac{2}{3c^3}\int\frac{dx}{c^3\pm x^3}$

174. $\int \frac{dx}{(c^3\pm x^3)^{n+1}} = \frac{1}{3nc^3}\left[\frac{x}{(c^3\pm x^3)^n} + (3n-1)\int\frac{dx}{(c^3\pm x^3)^n}\right]$

175. $\int \frac{x}{c^3\pm x^3}dx = \frac{1}{6c}\ln\left|\frac{c^3\pm x^3}{(c\pm x)^3}\right| \pm \frac{1}{c\sqrt{3}}\arctan\frac{2x\mp c}{c\sqrt{3}}$

176. $\int \frac{x}{(c^3\pm x^3)^2}dx = \frac{x^2}{3c^3(c^3\pm x^3)} + \frac{1}{3c^3}\int\frac{x}{c^3\pm x^3}dx$

177. $\int \frac{x}{(c^3\pm x^3)^{n+1}}dx = \frac{1}{3nc^3}\left[\frac{x^2}{(c^3\pm x^3)^n} + (3n-2)\int\frac{x}{(c^3\pm x^3)^n}dx\right]$

178. $\int \frac{x^2}{c^3\pm x^3}dx = \pm\frac{1}{3}\ln|c^3\pm x^3|$

179. $\int \frac{x^2}{(c^3\pm x^3)^2}dx = \mp\frac{1}{3(c^3\pm x^3)}$

180. $\int \frac{x^2}{(c^3\pm x^3)^{n+1}}dx = \mp\frac{1}{3n(c^3\pm x^3)^n}$

181. $\int \frac{x^3}{c^3\pm x^3}dx = \pm x \mp c^3\int\frac{dx}{c^3\pm x^3}$

182. $\int \frac{x^3}{(c^3\pm x^3)^2}dx = \mp\frac{x}{3(c^3\pm x^3)} \pm \frac{1}{3}\int\frac{dx}{c^3\pm x^3}$

183. $\int \frac{x^3}{(c^3 \pm x^3)^n}\mathrm{d}x = \frac{x^4}{3c^3(n-1)(c^3 \pm x^3)^{n-1}} + \frac{3n-7}{3c^3(n-1)}\int \frac{x^3}{(c^3 \pm x^3)^{n-1}}\mathrm{d}x$
$(n \neq 1)$

184. $\int \frac{x^m}{(c^3 \pm x^3)^n}\mathrm{d}x = \frac{x^{m+1}}{3c^3(n-1)(c^3 \pm x^3)^{n-1}} - \frac{m-3n+4}{3c^3(n-1)}\int \frac{x^m}{(c^3 \pm x^3)^{n-1}}\mathrm{d}x$
$(n \neq 1)$

185. $\int \frac{\mathrm{d}x}{x(c^3 \pm x^3)} = \frac{1}{3c^3}\ln\left|\frac{x^3}{c^3 \pm x^3}\right|$

186. $\int \frac{\mathrm{d}x}{x(c^3 \pm x^3)^2} = \frac{1}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^6}\ln\left|\frac{x^3}{c^3 \pm x^3}\right|$

187. $\int \frac{\mathrm{d}x}{x(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3(c^3 \pm x^3)^n} + \frac{1}{c^3}\int \frac{\mathrm{d}x}{x(c^3 \pm x^3)^n}$

188. $\int \frac{\mathrm{d}x}{x^2(c^3 \pm x^3)} = -\frac{1}{c^3 x} \mp \frac{1}{c^3}\int \frac{x}{c^3 \pm x^3}\mathrm{d}x$

189. $\int \frac{\mathrm{d}x}{x^2(c^3 \pm x^3)^2} = -\frac{1}{c^6 x} \mp \frac{x^2}{3c^6(c^3 \pm x^3)} \mp \frac{4}{3c^6}\int \frac{x}{c^3 \pm x^3}\mathrm{d}x$

190. $\int \frac{\mathrm{d}x}{x^2(c^3 \pm x^3)^{n+1}} = \frac{1}{c^3}\int \frac{\mathrm{d}x}{x^2(c^3 \pm x^3)^n} \mp \frac{1}{c^3}\int \frac{x}{(c^3 \pm x^3)^{n+1}}\mathrm{d}x$

191. $\int \frac{\mathrm{d}x}{x^m(c^3 \pm x^3)^n} = \frac{1}{3c^3(n-1)x^{m-1}(c^3 \pm x^3)^{n-1}} + \frac{m+3n-4}{3c^3(n-1)}\int \frac{\mathrm{d}x}{x^m(c^3 \pm x^3)^{n-1}}$
$(n \neq 1)$

192. $\int \frac{a+x}{a^3+x^3}\mathrm{d}x = \frac{2}{a\sqrt{3}}\arctan\frac{2x-a}{a\sqrt{3}}$

193. $\int \frac{a+x}{a^3-x^3}\mathrm{d}x = -\frac{1}{3a}\ln\frac{(a-x)^2}{a^2+ax+x^2}$

194. $\int \frac{a-x}{a^3+x^3}\mathrm{d}x = \frac{1}{3a}\ln\frac{(a+x)^2}{a^2-ax+x^2}$

195. $\int \frac{a-x}{a^3-x^3}\mathrm{d}x = \frac{2}{a\sqrt{3}}\arctan\frac{2x+a}{a\sqrt{3}}$

Ⅰ.1.2.8 含有 $c^4 + x^4$ 的积分

196. $\int \frac{\mathrm{d}x}{c^4+x^4} = \frac{1}{2c^3\sqrt{2}}\left(\frac{1}{2}\ln\frac{x^2+cx\sqrt{2}+c^2}{x^2-cx\sqrt{2}+c^2} + \arctan\frac{cx\sqrt{2}}{c^2-x^2}\right)$

197. $\int \frac{\mathrm{d}x}{(c^4+x^4)^2}$

$$= \frac{x}{4c^4(c^4+x^4)} + \frac{3}{16c^7\sqrt{2}}\ln\frac{x^2+cx\sqrt{2}+c^2}{x^2-cx\sqrt{2}+c^2} + \frac{3}{8c^7\sqrt{2}}\arctan\frac{cx\sqrt{2}}{c^2-x^2}$$

198. $\int \frac{\mathrm{d}x}{(c^4+x^4)^n} = \frac{x}{4(n-1)c^4(c^4+x^4)^{n-1}} + \frac{4n-5}{4(n-1)c^4}\int \frac{\mathrm{d}x}{(c^4+x^4)^{n-1}} \quad (n \neq 1)$

199. $\int \frac{x}{c^4+x^4}\mathrm{d}x = \frac{1}{2c^2}\arctan\frac{x^2}{c^2}$

200. $\int \frac{x^2}{c^4+x^4}\mathrm{d}x = \frac{1}{2c\sqrt{2}}\left(\frac{1}{2}\ln\frac{x^2-cx\sqrt{2}+c^2}{x^2+cx\sqrt{2}+c^2}+\arctan\frac{cx\sqrt{2}}{c^2-x^2}\right)$

201. $\int \frac{x^3}{c^4+x^4}\mathrm{d}x = \frac{1}{4}\ln(c^4+x^4)$

202. $\int \frac{x^m}{c^4+x^4}\mathrm{d}x = \frac{x^{m-3}}{m-3} - c^4\int\frac{x^{m-4}}{c^4+x^4}\mathrm{d}x \quad (m\neq 3)$

203. $\int \frac{x^2}{(c^4+x^4)^2}\mathrm{d}x$

$$= \frac{x^3}{4c^4(c^4+x^4)} + \frac{1}{16c^5\sqrt{2}}\ln\frac{x^2+cx\sqrt{2}+c^2}{x^2-cx\sqrt{2}+c^2} + \frac{3}{8c^5\sqrt{2}}\arctan\frac{cx\sqrt{2}}{c^2-x^2}$$

204. $\int \frac{x^m}{(c^4+x^4)^n}\mathrm{d}x = \frac{x^{m+1}}{4c^4(n-1)(c^4+x^4)^{n-1}} + \frac{4n-m-5}{4c^4(n-1)}\int\frac{x^m}{(c^4+x^4)^{n-1}}\mathrm{d}x$

$(n\neq 1)$

205. $\int \frac{\mathrm{d}x}{x(c^4+x^4)} = \frac{1}{2c^4}\ln\frac{x^2}{\sqrt{c^4+x^4}}$

206. $\int \frac{\mathrm{d}x}{x^2(c^4+x^4)} = -\frac{1}{c^4x} + \frac{1}{2c^5\sqrt{2}}\left(\arctan\frac{cx\sqrt{2}}{c^2+x^2} - \operatorname{artanh}\frac{cx\sqrt{2}}{c^2-x^2}\right)$

207. $\int \frac{\mathrm{d}x}{x^2(c^4+x^4)^2} = -\frac{1}{c^8x} - \frac{x^3}{4c^8(c^4+x^4)} - \frac{5}{4c^8}\int\frac{x^2}{c^4+x^4}\mathrm{d}x$

208. $\int \frac{\mathrm{d}x}{x^m(c^4+x^4)^n}$

$$= \frac{1}{4(n-1)c^4x^{m-1}(c^4+x^4)^{n-1}} + \frac{m+4n-5}{4(n-1)c^4}\int\frac{\mathrm{d}x}{x^m(c^4+x^4)^{n-1}} \quad (n\neq 1)$$

$$= -\frac{1}{4(m-1)c^4x^{m-1}(c^4+x^4)^{n-1}} - \frac{m+4n-5}{(m-1)c^4}\int\frac{\mathrm{d}x}{x^{m-4}(c^4+x^4)^n} \quad (m\neq 1)$$

Ⅰ.1.2.9 含有 c^4-x^4 的积分

209. $\int \frac{\mathrm{d}x}{c^4-x^4} = \frac{1}{2c^3}\left(\frac{1}{2}\ln\left|\frac{c+x}{c-x}\right| + \arctan\frac{x}{c}\right)$

210. $\int \frac{\mathrm{d}x}{(c^4-x^4)^2} = \frac{x}{4c^4(c^4-x^4)} + \frac{3}{16c^7}\ln\left|\frac{c+x}{c-x}\right| + \frac{3}{8c^7}\arctan\frac{x}{c}$

211. $\int \frac{\mathrm{d}x}{(c^4-x^4)^n} = \frac{x}{4(n-1)c^4(c^4-x^4)^{n-1}} + \frac{4n-5}{4(n-1)c^4}\int\frac{\mathrm{d}x}{(c^4-x^4)^{n-1}} \quad (n\neq 1)$

212. $\int \frac{x}{c^4-x^4}\mathrm{d}x = \frac{1}{4c^2}\ln\left|\frac{c^2+x^2}{c^2-x^2}\right|$

213. $\int \frac{x^2}{c^4-x^4}\mathrm{d}x = \frac{1}{2c}\left(\frac{1}{2}\ln\left|\frac{c+x}{c-x}\right| - \arctan\frac{x}{c}\right)$

214. $\int \frac{x^3}{c^4 - x^4}\mathrm{d}x = -\frac{1}{4}\ln|c^4 - x^4|$

215. $\int \frac{x^m}{c^4 - x^4}\mathrm{d}x = -\frac{x^{m-3}}{m-3} + c^4\int \frac{x^{m-4}}{c^4 - x^4}\mathrm{d}x \quad (m \neq 3)$

216. $\int \frac{x^2}{(c^4 - x^4)^2}\mathrm{d}x = \frac{x^2}{4c^4(c^4 - x^4)} - \frac{1}{16c^5}\ln\left|\frac{c-x}{c+x}\right| - \frac{1}{8c^5}\arctan\frac{x}{c}$

217. $\int \frac{x^m}{(c^4 - x^4)^n}\mathrm{d}x = \frac{x^{m+1}}{4c^4(n-1)(c^4 - x^4)^{n-1}} + \frac{4n - m - 5}{4c^4(n-1)}\int \frac{x^m}{(c^4 - x^4)^{n-1}}\mathrm{d}x$

$(n \neq 1)$

218. $\int \frac{\mathrm{d}x}{x(c^4 - x^4)} = \frac{1}{2c^4}\ln\frac{x^2}{\sqrt{c^4 - x^4}}$

219. $\int \frac{\mathrm{d}x}{x^2(c^4 - x^4)} = -\frac{1}{c^4 x} + \frac{1}{2c^5}\left(\ln\sqrt{\frac{c+x}{c-x}} - \arctan\frac{x}{c}\right)$

220. $\int \frac{\mathrm{d}x}{x^2(c^4 - x^4)^2} = -\frac{1}{c^8 x} + \frac{x^3}{4c^8(c^4 - x^4)} + \frac{5}{4c^8}\int \frac{x^2}{c^4 - x^4}\mathrm{d}x$

221. $\int \frac{\mathrm{d}x}{x^m(c^4 - x^4)^n}$

$$= \frac{1}{4(n-1)c^4 x^{m-1}(c^4 - x^4)^{n-1}} + \frac{m + 4n - 5}{4(n-1)c^4}\int \frac{\mathrm{d}x}{x^m(c^4 - x^4)^{n-1}} \quad (n \neq 1)$$

$$= -\frac{1}{(m-1)c^4 x^{m-1}(c^4 - x^4)^{n-1}} + \frac{m + 4n - 5}{(m-1)c^4}\int \frac{\mathrm{d}x}{x^{m-4}(c^4 - x^4)^n} \quad (m \neq 1)$$

Ⅰ.1.2.10 含有 $a + bx + cx^2$ 的积分

设 $X = a + bx + cx^2$ 和 $q = 4ac - b^2$，$q \neq 0$. $\left[\text{如果 } q = 0\text{,则 } X = c\left(x + \frac{b}{2c}\right)^2\text{,}\right.$ 这是 Ⅰ.1.2.1 小节的情形,应该使用其中相应的公式.]

222. $\int X^2\mathrm{d}x = \frac{(b + 2cx)q^2}{60c^3}\left(1 + \frac{2cX}{q} + \frac{12c^2X^2}{2q^2}\right) \quad (q \neq 0)$

223. $\int X^n\mathrm{d}x = (-1)^n\frac{(n!)^2}{(2n+1)!}\frac{b + 2cx}{2c}\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\left(\frac{b^2 - 4ac}{c}\right)^{n-k}X^k \quad (q \neq 0)$

224. $\int xX^n\mathrm{d}x = \frac{X^{n+1}}{2(n+1)c} - \frac{b}{2(n+1)c} - \frac{b}{2c}\int X^n\mathrm{d}x$

225. $\int \frac{\mathrm{d}x}{X} = \begin{cases} \frac{2}{\sqrt{q}}\arctan\frac{2cx + b}{\sqrt{q}} & (q > 0) \\ -\frac{2}{\sqrt{-q}}\operatorname{artanh}\frac{2cx + b}{\sqrt{-q}} & (q < 0) \\ \frac{1}{\sqrt{-q}}\ln\left|\frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}\right| & (q < 0) \end{cases}$

226. $\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q}\int \frac{dx}{X}$

227. $\int \frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{nq}\int \frac{dx}{X^n}$

$$= \frac{(2n)!}{(n!)^2}\left(\frac{c}{q}\right)^n\left[\frac{2cx+b}{q}\sum_{r=1}^{n}\left(\frac{q}{cX}\right)^r\frac{(r-1)!r!}{(2r)!} + \int\frac{dx}{X}\right]$$

228. $\int \frac{x}{X}dx = \frac{1}{2c}\ln|X| - \frac{b}{2c}\int\frac{dx}{X}$

229. $\int \frac{x}{X^2}dx = -\frac{2a+bx}{qX} - \frac{b}{q}\int\frac{dx}{X}$

230. $\int \frac{x}{X^{n+1}}dx = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq}\int\frac{dx}{X^n} \quad (n \neq 0)$

231. $\int \frac{x^2}{X}dx = \frac{x}{c} - \frac{b}{2c^2}\ln|X| + \frac{b^2-2ac}{2c^2}\int\frac{dx}{X}$

232. $\int \frac{x^2}{X^2}dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q}\int\frac{dx}{X}$

233. $\int \frac{x^m}{X^{n+1}}dx = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1}\frac{b}{c}\int\frac{x^{m-1}}{X^{n+1}}dx$

$$+\frac{m-1}{2n-m+1}\frac{a}{c}\int\frac{x^{m-2}}{X^{n+1}}dx$$

234. $\int \frac{dx}{xX} = \frac{1}{2a}\ln\frac{x^2}{|X|} - \frac{b}{2a}\int\frac{dx}{X}$

235. $\int \frac{dx}{x^2X} = \frac{b}{2a^2}\ln\frac{|X|}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right)\int\frac{dx}{X}$

236. $\int \frac{dx}{x^2X^2} = -\frac{b}{a^3}\ln\frac{x^2}{|X|} - \frac{a+bx}{a^2xX} + \frac{(b^2-3ac)(b+2cx)}{a^2qX}$

$$-\frac{1}{q}\left(\frac{b^4}{a^3} - \frac{6b^2c}{a^2} + \frac{6c^2}{a}\right)\int\frac{dx}{X}$$

237. $\int \frac{dx}{x^mX^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\frac{b}{a}\int\frac{dx}{x^{m-1}X^{n+1}}$

$$-\frac{2n+m-1}{m-1}\frac{c}{a}\int\frac{dx}{x^{m-2}X^{n+1}} \quad (m \neq 1)$$

Ⅰ.1.2.11 含有 $a+bx^k$ 和 $\sqrt{x}$ 的积分

238. $\int \frac{dx}{(a+bx)\sqrt{x}} = \begin{cases} \frac{2}{\sqrt{ab}}\arctan\sqrt{\frac{bx}{a}} & (ab>0) \\ \frac{1}{i\sqrt{ab}}\ln\frac{a-bx+2i\sqrt{abx}}{a+bx} & (ab<0) \end{cases}$

239. $\int \frac{\mathrm{d}x}{(a+bx)^2\sqrt{x}} = \frac{\sqrt{x}}{a(a+bx)} + \frac{1}{2a}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

240. $\int \frac{\sqrt{x}}{a+bx}\mathrm{d}x = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

241. $\int \frac{x\sqrt{x}}{a+bx}\mathrm{d}x = 2\sqrt{x}\left(\frac{x}{3b} - \frac{a}{b^2}\right) + \frac{a^2}{b^2}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

242. $\int \frac{x^2\sqrt{x}}{a+bx}\mathrm{d}x = 2\sqrt{x}\left(\frac{x^2}{5b} - \frac{ax}{3b^2} + \frac{a^2}{b^3}\right) - \frac{a^3}{b^3}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

243. $\int \frac{x^m\sqrt{x}}{a+bx}\mathrm{d}x = 2\sqrt{x}\sum_{k=0}^{m} \frac{(-1)^k a^k x^{m-k}}{(2m-2k+1)b^{k+1}} + (-1)^{m+1}\frac{a^{m+1}}{b^{m+1}}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

244. $\int \frac{\sqrt{x}}{(a+bx)^2}\mathrm{d}x = -\frac{\sqrt{x}}{b(a+bx)} + \frac{1}{2b}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

245. $\int \frac{x\sqrt{x}}{(a+bx)^2}\mathrm{d}x = \frac{2x\sqrt{x}}{b(a+bx)} - \frac{3a}{b}\int \frac{\sqrt{x}}{(a+bx)^2}\mathrm{d}x$

246. $\int \frac{x^2\sqrt{x}}{(a+bx)^2}\mathrm{d}x = \frac{2\sqrt{x}}{a+bx}\left(\frac{x^2}{3b} - \frac{5ax}{3b^2}\right) + \frac{5a^2}{b^2}\int \frac{\sqrt{x}}{(a+bx)^2}\mathrm{d}x$

247. $\int \frac{\sqrt{x}}{(a+bx)^3}\mathrm{d}x = \sqrt{x}\left[\frac{1}{4ab(a+bx)} - \frac{1}{2b(a+bx)^2}\right] + \frac{1}{8ab}\int \frac{\mathrm{d}x}{(a+bx)\sqrt{x}}$

248. $\int \frac{x\sqrt{x}}{(a+bx)^3}\mathrm{d}x = -\frac{2x\sqrt{x}}{b(a+bx)^2} + \frac{3a}{b}\int \frac{\sqrt{x}}{(a+bx)^3}\mathrm{d}x$

249. $\int \frac{x^2\sqrt{x}}{(a+bx)^3}\mathrm{d}x = \frac{2\sqrt{x}}{(a+bx)^2}\left(\frac{x^2}{b} + \frac{5ax}{b^2}\right) - \frac{15a^2}{b^2}\int \frac{\sqrt{x}}{(a+bx)^3}\mathrm{d}x$

250. $\int \frac{\sqrt{x}}{a+bx^2}\mathrm{d}x = \begin{cases} \frac{1}{b\alpha\sqrt{2}}\left(-\ln\left|\frac{x+\alpha\sqrt{2x}+\alpha^2}{\sqrt{a+bx^2}}\right| + \arctan\frac{\alpha\sqrt{2x}}{\alpha^2-x}\right) & \left(\frac{a}{b}>0\right) \\ \frac{1}{2b\beta}\left(\ln\left|\frac{\beta-\sqrt{x}}{\beta+\sqrt{x}}\right| + 2\arctan\frac{\sqrt{x}}{\beta}\right) & \left(\frac{a}{b}<0\right) \end{cases}$

$\left(\text{这里，}\alpha = \sqrt[4]{\frac{a}{b}},\ \beta = \sqrt[4]{-\frac{a}{b}}\right)$

251. $\int \frac{x\sqrt{x}}{a+bx^2}\mathrm{d}x = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int \frac{\mathrm{d}x}{(a+bx^2)\sqrt{x}}$

252. $\int \frac{x^2\sqrt{x}}{a+bx^2}\mathrm{d}x = \frac{2x\sqrt{x}}{3b} - \frac{a}{b}\int \frac{\sqrt{x}}{a+bx^2}\mathrm{d}x$

253. $\int \frac{\sqrt{x}}{(a+bx^2)^2}\mathrm{d}x = \frac{x\sqrt{x}}{2a(a+bx^2)} + \frac{1}{4a}\int \frac{\sqrt{x}}{a+bx^2}\mathrm{d}x$

254. $\int \frac{x\sqrt{x}}{(a+bx^2)^2}\mathrm{d}x = -\frac{\sqrt{x}}{2b(a+bx^2)} + \frac{1}{4b}\int \frac{\mathrm{d}x}{(a+bx^2)\sqrt{x}}$

255. $\int \frac{x^2\sqrt{x}}{(a+bx^2)^2}\mathrm{d}x = -\frac{x\sqrt{x}}{2b(a+bx^2)} + \frac{3}{4b}\int \frac{\sqrt{x}}{a+bx^2}\mathrm{d}x$

256. $\int \frac{\mathrm{d}x}{(a+bx^2)\sqrt{x}} = \begin{cases} \frac{1}{b\alpha^3\sqrt{2}}\left(\ln\frac{x+\alpha\sqrt{2x}+\alpha^2}{\sqrt{a+bx^2}} + \arctan\frac{\alpha\sqrt{2x}}{\alpha^2-x}\right) & \left(\frac{a}{b}>0\right) \\ \frac{1}{2b\beta^3}\left(\ln\left|\frac{\beta-\sqrt{x}}{\beta+\sqrt{x}}\right| - 2\arctan\frac{\sqrt{x}}{\beta}\right) & \left(\frac{a}{b}<0\right) \end{cases}$

$\left(\text{这里}, \alpha = \sqrt[4]{\frac{a}{b}},\ \beta = \sqrt[4]{-\frac{a}{b}}\right)$

257. $\int \frac{\mathrm{d}x}{(a+bx^2)^2\sqrt{x}} = \frac{\sqrt{x}}{2a(a+bx)} + \frac{3}{4a}\int \frac{\mathrm{d}x}{(a+bx^2)\sqrt{x}}$

Ⅰ.1.2.12 含有 $\sqrt{a+bx}$ 和 $\alpha+\beta x$ 的积分

设 $z = a+bx$，$t = \alpha+\beta x$ 和 $\Delta = a\beta - b\alpha$.

258. $\int \frac{t}{\sqrt{z}}\mathrm{d}x = \frac{2\alpha\sqrt{z}}{b} + \beta\left(\frac{z}{3}-a\right)\frac{2\sqrt{z}}{b^2}$

259. $\int \frac{t^2}{\sqrt{z}}\mathrm{d}x = \frac{2\alpha^2\sqrt{z}}{b} + 2\alpha\beta\left(\frac{z}{3}-a\right)\frac{2\sqrt{z}}{b^2} + \beta^2\left(\frac{z^2}{5}-\frac{2az}{3}+a^2\right)\frac{2\sqrt{z}}{b^3}$

260. $\int \frac{tz}{\sqrt{z}}\mathrm{d}x = \frac{2\alpha\sqrt{z^3}}{3b} + \beta\left(\frac{z}{5}-\frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2}$

261. $\int \frac{t^2z}{\sqrt{z}}\mathrm{d}x = \frac{2\alpha^2\sqrt{z^3}}{3b} + 2\alpha\beta\left(\frac{z}{5}-\frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2} + \beta^2\left(\frac{z^2}{7}-\frac{2az}{5}+\frac{a^2}{3}\right)\frac{2\sqrt{z^3}}{b^3}$

262. $\int \frac{tz^2}{\sqrt{z}}\mathrm{d}x = \frac{2\alpha\sqrt{z^5}}{5b} + \beta\left(\frac{z}{7}-\frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2}$

263. $\int \frac{t^2z^2}{\sqrt{z}}\mathrm{d}x = \frac{2\alpha^2\sqrt{z^5}}{5b} + 2\alpha\beta\left(\frac{z}{7}-\frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2} + \beta^2\left(\frac{z^2}{9}-\frac{2az}{7}+\frac{a^2}{5}\right)\frac{2\sqrt{z^5}}{b^3}$

264. $\int \frac{t^nz^m}{\sqrt{z}}\mathrm{d}x = 2\sqrt{z^{2m+1}}\sum_{k=0}^{n}\left[\binom{n}{k}\frac{\alpha^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2m+2k-2p+1}\right]$

265. $\int \frac{t}{z\sqrt{z}}\mathrm{d}x = -\frac{2\alpha}{b\sqrt{z}} + \frac{2\beta(z+a)}{b^2\sqrt{z}}$

266. $\int \frac{t^2}{z\sqrt{z}}\mathrm{d}x = -\frac{2\alpha^2}{b\sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2\sqrt{z}} + \frac{2\beta^2(z^2-6az-3a^2)}{3b^3\sqrt{z}}$

267. $\int \frac{t}{z^2\sqrt{z}}\mathrm{d}x = -\frac{2\alpha}{3b\sqrt{z^3}} - \frac{2\beta(3z-a)}{3b^2\sqrt{z^3}}$

268. $\int \frac{t^2}{z^2\sqrt{z}}\mathrm{d}x = -\frac{2\alpha^2}{3b\sqrt{z^3}} - \frac{4\alpha\beta(3z-a)}{3b^2\sqrt{z^3}} + \frac{2\beta^2(3z^2+6az-a^2)}{3b^3\sqrt{z^3}}$

269. $\int \frac{t^n}{z^m\sqrt{z}}\mathrm{d}x = \frac{2}{\sqrt{z^{2m-1}}}\sum_{k=0}^{n}\left[\binom{n}{k}\frac{\alpha^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2k-2p-2m+1}\right]$

270. $\int \frac{z}{t\sqrt{z}}\mathrm{d}x = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

271. $\int \frac{z^2}{t\sqrt{z}}\mathrm{d}x = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

272. $\int \frac{z}{t^2\sqrt{z}}\mathrm{d}x = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

273. $\int \frac{z^2}{t^2\sqrt{z}}\mathrm{d}x = -\frac{z^2\sqrt{z}}{\Delta t} + \frac{bz\sqrt{z}}{\beta\Delta} + \frac{3b\sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

274. $\int \frac{z^m}{t^n\sqrt{z}}\mathrm{d}x = -\frac{2}{(2n-2m-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta}\int\frac{z^{m-1}}{t^n\sqrt{z}}\mathrm{d}x$

$$= -\frac{1}{(n-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta}\int\frac{z^{m-1}}{t^{n-1}\sqrt{z}}\mathrm{d}x$$

275. $\int \frac{\mathrm{d}x}{t\sqrt{z}} = \begin{cases} \frac{1}{\sqrt{\beta\Delta}}\ln\left|\frac{\beta\sqrt{z}-\sqrt{\beta\Delta}}{\beta\sqrt{z}+\sqrt{\beta\Delta}}\right| & (\beta\Delta > 0) \\ \frac{1}{\sqrt{-\beta\Delta}}\arctan\frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}} & (\beta\Delta < 0) \\ -\frac{2\sqrt{z}}{bt} & (\Delta = 0) \end{cases}$

276. $\int \frac{\mathrm{d}x}{t^2\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

277. $\int \frac{\mathrm{d}x}{tz\sqrt{z}} = \frac{2}{\Delta\sqrt{z}} + \frac{\beta}{\Delta}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

278. $\int \frac{\mathrm{d}x}{tz^2\sqrt{z}} = \frac{2}{3\Delta z\sqrt{z}} + \frac{2\beta}{\Delta^2\sqrt{z}} + \frac{\beta^2}{\Delta^2}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

279. $\int \frac{\mathrm{d}x}{t^2z\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

280. $\int \frac{\mathrm{d}x}{t^2z^2\sqrt{z}} = -\frac{1}{\Delta tz\sqrt{z}} - \frac{5b}{3\Delta^2z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3}\int\frac{\mathrm{d}x}{t\sqrt{z}}$

281. $\int \frac{\mathrm{d}x}{z^mt^n\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{\sqrt{z}}{z^mt^{n-1}} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta}\int\frac{\mathrm{d}x}{z^{m-1}t^n\sqrt{z}}$

$$= -\frac{1}{(n-1)\Delta}\frac{\sqrt{z}}{z^mt^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta}\int\frac{\mathrm{d}x}{z^mt^{n-1}\sqrt{z}}$$

Ⅰ.1.2.13 含有 $\sqrt{a+bx}$ 和 $\sqrt{c+dx}$ 的积分

设 $u=a+bx$，$v=c+dx$ 和 $k=ad-bc$，$k\neq 0$.（如果 $k=0$，那么 $v=\frac{c}{a}u$，应该使用另外的公式.）

282. $\int \frac{\mathrm{d}x}{\sqrt{uv}}=\begin{cases}\frac{2}{\sqrt{bd}}\ln(\sqrt{du}+\sqrt{bv}) & (bd>0)\\ \frac{2}{\sqrt{-bd}}\arctan\sqrt{-\frac{du}{bv}} & (bd<0)\end{cases}$

283. $\int \sqrt{uv}\mathrm{d}x=\frac{k+2bv}{4bd}\sqrt{uv}-\frac{k^2}{8bd}\int\frac{\mathrm{d}x}{\sqrt{uv}}$

284. $\int v^m\sqrt{u}\mathrm{d}x=\frac{1}{(2m+3)d}\left(2v^{m+1}\sqrt{u}+k\int\frac{v^m}{\sqrt{u}}\mathrm{d}x\right)$

285. $\int\frac{\sqrt{u}}{v}\mathrm{d}x=\frac{2\sqrt{u}}{d}+\frac{k}{d}\int\frac{\mathrm{d}x}{v\sqrt{u}}$

286. $\int\frac{\sqrt{u}}{v^n}\mathrm{d}x=-\frac{1}{(n-1)d}\left(\frac{\sqrt{u}}{v^{n-1}}-\frac{b}{2}\int\frac{\mathrm{d}x}{v^{n-1}\sqrt{u}}\right)\quad(n\neq 1)$

287. $\int\frac{v}{\sqrt{u}}\mathrm{d}x=\frac{2\sqrt{u}}{3b}\left(v-\frac{2k}{b}\right)$

288. $\int\frac{v^m}{\sqrt{u}}\mathrm{d}x=\frac{2}{b(2m+1)}\left(v^m\sqrt{u}-mk\int\frac{v^{m-1}}{\sqrt{u}}\mathrm{d}x\right)$

$$=\frac{2(m!)^2\sqrt{u}}{b(2m+1)!}\sum_{r=0}^{m}\left(-\frac{4k}{b}\right)^{m-r}\frac{(2r)!}{(r!)^2}v^r$$

289. $\int\frac{x}{\sqrt{uv}}\mathrm{d}x=\frac{\sqrt{uv}}{bd}-\frac{ad+bc}{2bd}\int\frac{\mathrm{d}x}{\sqrt{uv}}$

290. $\int\frac{v}{\sqrt{uv}}\mathrm{d}x=\frac{\sqrt{uv}}{b}-\frac{k}{2b}\int\frac{\mathrm{d}x}{\sqrt{uv}}$

291. $\int\sqrt{\frac{v}{u}}\mathrm{d}x=\frac{v}{|v|}\int\frac{v}{\sqrt{uv}}\mathrm{d}x$

292. $\int\frac{\mathrm{d}x}{v\sqrt{uv}}=-\frac{2\sqrt{uv}}{kv}$

293. $$\int \frac{\mathrm{d}x}{v\sqrt{u}} = \begin{cases} \dfrac{1}{\sqrt{kd}}\ln\left|\dfrac{d\sqrt{u}-\sqrt{kd}}{d\sqrt{u}+\sqrt{kd}}\right| & (kd>0) \\ \dfrac{1}{\sqrt{kd}}\ln\dfrac{(d\sqrt{u}-\sqrt{kd})^2}{|v|} & (kd>0) \\ \dfrac{2}{\sqrt{-kd}}\arctan\dfrac{d\sqrt{u}}{\sqrt{-kd}} & (kd<0) \end{cases}$$

294. $$\int \frac{\mathrm{d}x}{v^m\sqrt{u}} = -\frac{1}{(m-1)k}\left[\frac{\sqrt{u}}{v^{m-1}} + \left(m-\frac{3}{2}\right)b\int\frac{\mathrm{d}x}{v^{m-1}\sqrt{u}}\right] \quad (m\neq 1)$$

Ⅰ.1.2.14 含有 $\sqrt{a+bx}$ 和 $\sqrt[p]{(a+bx)^n}$ 的积分

295. $$\int\sqrt{a+bx}\,\mathrm{d}x = \frac{2}{3b}\sqrt{(a+bx)^3}$$

296. $$\int\sqrt{(a+bx)^n}\,\mathrm{d}x = \frac{2}{(n+2)b}\sqrt{(a+bx)^{n+2}}$$

297. $$\int\sqrt[p]{(a+bx)^n}\,\mathrm{d}x = \frac{p}{(n+p)b}\sqrt[p]{(a+bx)^{n+p}}$$

298. $$\int x\sqrt{a+bx}\,\mathrm{d}x = -\frac{2(2a-3bx)}{15b^2}\sqrt{(a+bx)^3}$$

299. $$\int x\sqrt{(a+bx)^n}\,\mathrm{d}x = \frac{2x}{(n+2)b}\left[1-\frac{2(a+bx)}{(n+4)bx}\right]\sqrt{(a+bx)^{n+2}}$$

300. $$\int x\sqrt[p]{(a+x)^n}\,\mathrm{d}x = \frac{px}{(n+p)b}\left[1-\frac{p(a+bx)}{(n+2p)bx}\right]\sqrt[p]{(a+bx)^{n+p}}$$

301. $$\int x^2\sqrt{a+bx}\,\mathrm{d}x = \frac{2(8a^2-12abx+15b^2x^2)}{105b^3}\sqrt{(a+bx)^3}$$

302. $$\int x^2\sqrt{(a+bx)^n}\,\mathrm{d}x$$
$$= \frac{2x^2}{(n+2)b}\left\{1-\frac{4(a+bx)}{(n+4)bx}\left[1-\frac{2(a+bx)}{(n+6)bx}\right]\right\}\sqrt{(a+bx)^{n+2}}$$

303. $$\int x^2\sqrt[p]{(a+bx)^n}\,\mathrm{d}x$$
$$= \frac{px^2}{(n+p)b}\left\{1-\frac{2p(a+bx)}{(n+2p)bx}\left[1-\frac{p(a+bx)}{(n+3p)bx}\right]\right\}\sqrt[p]{(a+bx)^{n+p}}$$

304. $$\int x^m\sqrt[p]{(a+bx)^n}\,\mathrm{d}x$$
$$= \frac{p}{b^{m+1}}(a+bx)^{1+\frac{n}{p}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{a^k}{(m-k+1)p+n}(a+bx)^{m-k}$$

305. $$\int\frac{\sqrt{a+bx}}{x}\,\mathrm{d}x = 2\sqrt{a+bx}+a\int\frac{\mathrm{d}x}{x\sqrt{a+bx}}$$

306. $\int \frac{\sqrt{a+bx}}{x^2}\mathrm{d}x = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2}\int \frac{\mathrm{d}x}{x\sqrt{a+bx}}$

307. $\int \frac{\sqrt{a+bx}}{x^m}\mathrm{d}x = -\frac{1}{(m-1)a}\left[\frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2}\int \frac{\sqrt{a+bx}}{x^{m-1}}\mathrm{d}x\right]$

308. $\int \frac{\sqrt{(a+bx)^n}}{x}\mathrm{d}x = \frac{2\sqrt{(a+bx)^n}}{n} + a\int \frac{\sqrt{(a+bx)^{n-2}}}{x}\mathrm{d}x$

309. $\int \frac{\sqrt{(a+bx)^n}}{x^2}\mathrm{d}x = -\frac{\sqrt{(a+bx)^{n+2}}}{ax} + \frac{nb}{2a}\int \frac{\sqrt{(a+bx)^n}}{x}\mathrm{d}x$

310. $\int \frac{\sqrt{(a+bx)^n}}{x^m}\mathrm{d}x = -\frac{\sqrt{(a+bx)^{n+2}}}{a(m-1)x^{m-1}} - \frac{b(2m-n-4)}{2a(m-1)}\int \frac{\sqrt{(a+bx)^n}}{x^{m-1}}\mathrm{d}x$

$(m \neq 1)$

311. $\int \frac{\mathrm{d}x}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$

312. $\int \frac{\mathrm{d}x}{\sqrt{(a+bx)^n}} = -\frac{2}{(n-2)b\sqrt{(a+bx)^{n-2}}} \quad (n \neq 2)$

313. $\int \frac{\mathrm{d}x}{\sqrt[p]{(a+bx)^n}} = \frac{p}{(p-n)b\sqrt[p]{(a+bx)^{n-p}}} \quad (p \neq n)$

314. $\int \frac{x}{\sqrt{a+bx}}\mathrm{d}x = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}$

315. $\int \frac{x}{\sqrt{(a+bx)^n}}\mathrm{d}x = -\frac{2x}{(n-2)b\sqrt{(a+bx)^{n-2}}}\left[1 - \frac{2(a+bx)}{(4-n)bx}\right] \quad (n \neq 2, 4)$

316. $\int \frac{x}{\sqrt[p]{(a+bx)^n}}\mathrm{d}x = \frac{px}{(p-n)b\sqrt[p]{(a+bx)^{n-p}}}\left[1 - \frac{p(a+bx)}{(2p-n)bx}\right]$

$(n \neq p, 2p)$

317. $\int \frac{x^2}{\sqrt{a+bx}}\mathrm{d}x = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}$

318. $\int \frac{x^2}{\sqrt{(a+bx)^n}}\mathrm{d}x$

$$= -\frac{2x^2}{(n-2)b\sqrt{(a+bx)^{n-2}}}\left\{1 - \frac{4(a+bx)}{(4-n)bx}\left[1 - \frac{2(a+bx)}{(6-n)bx}\right]\right\}$$

$(n \neq 2, 4, 6)$

319. $\int \frac{x^2}{\sqrt[p]{(a+bx)^n}}\mathrm{d}x$

$$= -\frac{px^2}{(n-p)b\sqrt[p]{(a+bx)^{n-p}}}\left\{1 - \frac{2p(a+bx)}{(2p-n)bx}\left[1 - \frac{p(a+bx)}{(3p-n)bx}\right]\right\}$$

$(n \neq p, 2p, 3p)$

320. $\int \frac{x^3}{\sqrt[p]{(a+bx)^n}}\mathrm{d}x = \frac{px^3}{(p-n)b\sqrt[p]{(a+bx)^{n-p}}}$

$$\cdot\left(1-\frac{3p(a+bx)}{(2p-n)bx}\left\{1-\frac{2p(a+bx)}{(3p-n)bx}\left[1-\frac{p(a+bx)}{(4p-n)bx}\right]\right\}\right)$$

$(n \neq p, 2p, 3p, 4p)$

321. $\int \frac{x^m}{\sqrt[p]{(a+bx)^n}}\mathrm{d}x$

$$= \frac{p}{b^{m+1}\sqrt[p]{(a+bx)^{n-p}}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{a^k}{(m-k+1)p-n}(a+bx)^{m-k}$$

$[n \neq p, 2p, 3p, \cdots, (m+1)p]$

322. $\int \frac{\mathrm{d}x}{x\sqrt{a+bx}} = \begin{cases} \frac{2}{\sqrt{-a}}\arctan\sqrt{\frac{a+bx}{-a}} & (a<0) \\ \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| & (a>0) \end{cases}$

323. $\int \frac{\mathrm{d}x}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{\mathrm{d}x}{x\sqrt{a+bx}}$

324. $\int \frac{\mathrm{d}x}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int\frac{\mathrm{d}x}{x^{n-1}\sqrt{a+bx}}$

$$= \frac{(2n-2)!}{[(n-1)!]^2}\left[-\frac{\sqrt{a+bx}}{a}\sum_{r=0}^{n-1}\frac{r!(r-1)!}{x^r(2r)!}\left(-\frac{b}{4a}\right)^{n-r-1}\right.$$

$$\left.+\left(-\frac{b}{4a}\right)^{n-1}\int\frac{\mathrm{d}x}{x\sqrt{a+bx}}\right]$$

325. $\int \frac{\mathrm{d}x}{x\sqrt{(a+bx)^n}} = \frac{2}{(n-2)a\sqrt{(a+bx)^{n-2}}} + \frac{1}{a}\int\frac{\mathrm{d}x}{x\sqrt{(a+bx)^{n-2}}}$

$(n>2)$

326. $\int \frac{\mathrm{d}x}{x^m\sqrt{(a+bx)^n}} = -\frac{1}{a(m-1)x^{m-1}\sqrt{(a+bx)^{n-2}}}$

$$-\frac{b(2m+n-4)}{2a(m-1)}\int\frac{\mathrm{d}x}{x^{m-1}\sqrt{(a+bx)^n}} \quad (m \neq 1)$$

327. $\int \sqrt{x(a+bx)}\mathrm{d}x = \frac{a+2bx}{4b}\sqrt{x(a+bx)} - \frac{a^2}{8b\sqrt{b}}\operatorname{arcosh}\frac{a+2bx}{a}$

328. $\int \sqrt{\frac{a+bx}{x}}\mathrm{d}x = \sqrt{x}(a+bx) + \frac{a}{\sqrt{b}}\ln|a+bx+\sqrt{bx}|$

329. $\int \sqrt{\frac{x}{a+bx}}\mathrm{d}x = \frac{\sqrt{x}}{b}(a+bx) - \frac{a}{b\sqrt{b}}\ln|a+bx+\sqrt{bx}|$

Ⅰ.1.2.15 含有 $\sqrt{x^2 \pm a^2}$ 的积分

330. $\int \sqrt{x^2 \pm a^2}\mathrm{d}x = \frac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|\right)$

331. $\int \sqrt{(x^2 \pm a^2)^3}\mathrm{d}x$

$$= \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right|\right]$$

332. $\int \sqrt{(x^2 \pm a^2)^n}\mathrm{d}x = \frac{1}{n+1}\left[x\sqrt{(x^2 \pm a^2)^n} \pm na^2\int \sqrt{(x^2 \pm a^2)^{n-2}}\mathrm{d}x\right]$

333. $\int x\sqrt{x^2 \pm a^2}\mathrm{d}x = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3}$

334. $\int x\sqrt{(x^2 \pm a^2)^3}\mathrm{d}x = \frac{1}{5}\sqrt{(x^2 \pm a^2)^5}$

335. $\int x\sqrt{(x^2 + a^2)^n}\mathrm{d}x = \frac{1}{n+2}\sqrt{(x^2 + a^2)^{n+2}}$

336. $\int x\sqrt{(x^2 - a^2)^n}\mathrm{d}x = \frac{1}{n+2}\left[x^2\sqrt{(x^2 - a^2)^n} - na^2\int x\sqrt{(x^2 - a^2)^{n-2}}\mathrm{d}x\right]$

337. $\int x^2\sqrt{x^2 \pm a^2}\mathrm{d}x = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2x}{8}\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left|x + \sqrt{x^2 \pm a^2}\right|$

338. $\int x^2\sqrt{(x^2 \pm a^2)^3}\mathrm{d}x = \frac{x}{6}\sqrt{(x^2 \pm a^2)^5} \mp \frac{a^2x}{24}\sqrt{(x^2 \pm a^2)^3}$

$$- \frac{a^4x}{16}\sqrt{x^2 \pm a^2} \mp \frac{a^6}{16}\ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

339. $\int x^2\sqrt{(x^2 + a^2)^n}\mathrm{d}x = \frac{x\sqrt{(x^2 + a^2)^{n+2}}}{n+3} - \frac{a^2}{n+3}\int \sqrt{(x^2 + a^2)^n}\mathrm{d}x$

340. $\int x^2\sqrt{(x^2 - a^2)^n}\mathrm{d}x = \frac{x^3\sqrt{(x^2 - a^2)^n}}{n+3} - \frac{na^2}{n+3}\int x^2\sqrt{(x^2 - a^2)^{n-2}}\mathrm{d}x$

341. $\int x^3\sqrt{(x^2 + a^2)^n}\mathrm{d}x = \left(x^2 - \frac{2a^2}{n+2}\right)\frac{\sqrt{(x^2 + a^2)^{n+2}}}{n+4} \quad (n \neq -2, -4)$

342. $\int x^3\sqrt{(x^2 - a^2)^n}\mathrm{d}x = \frac{x^4\sqrt{(x^2 - a^2)^n}}{n+4} - \frac{na^2}{n+4}\int x^3\sqrt{(x^2 - a^2)^{n-2}}\mathrm{d}x$

$(n \neq -4)$

343. $\int x^m\sqrt[p]{(x^2 + a^2)^n}\mathrm{d}x$

$$= \frac{px^{m-1}\sqrt[p]{(x^2 + a^2)^{n+p}}}{2n + mp + p} - \frac{(m-1)pa^2}{2n + mp + p}\int x^{m-2}\sqrt[p]{(x^2 + a^2)^n}\mathrm{d}x$$

344. $\int x^m\sqrt[p]{(x^2 - a^2)^n}\mathrm{d}x$

$$= \frac{px^{m+1}\sqrt[p]{(x^2-a^2)^n}}{2n+mp+p} - \frac{2na^2}{2n+mp+p}\int x^m\sqrt[p]{(x^2-a^2)^{n-p}}\,\mathrm{d}x$$

345. $\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln|x+\sqrt{x^2 \pm a^2}|$

346. $\int \frac{\mathrm{d}x}{\sqrt{(x^2 \pm a^2)^3}} = \pm \frac{x}{a^2\sqrt{x^2 \pm a^2}}$

347. $\int \frac{\mathrm{d}x}{\sqrt{(x^2 \pm a^2)^n}} = \pm \frac{x}{(n-2)a^2\sqrt{(x^2 \pm a^2)^{n-2}}} \pm \frac{n-3}{(n-2)a^2}\int \frac{\mathrm{d}x}{\sqrt{(x^2 \pm a^2)^{n-2}}}$

$(n \neq 2)$

348. $\int \frac{x}{\sqrt{x^2 \pm a^2}}\mathrm{d}x = \sqrt{x^2 \pm a^2}$

349. $\int \frac{x}{\sqrt{(x^2 \pm a^2)^3}}\mathrm{d}x = -\frac{1}{\sqrt{x^2 \pm a^2}}$

350. $\int \frac{x}{\sqrt{(x^2 \pm a^2)^n}}\mathrm{d}x = -\frac{1}{(n-2)\sqrt{(x^2 \pm a^2)^{n-2}}} \quad (n \neq 2)$

351. $\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\mathrm{d}x = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\ln|x+\sqrt{x^2 \pm a^2}|$

352. $\int \frac{x^2}{\sqrt{(x^2 \pm a^2)^3}}\mathrm{d}x = -\frac{x}{\sqrt{x^2 \pm a^2}} + \ln|x+\sqrt{x^2 \pm a^2}|$

353. $\int \frac{x^2}{\sqrt{(x^2 \pm a^2)^n}}\mathrm{d}x = \pm \frac{x^3}{(n-2)a^2\sqrt{(x^2 \pm a^2)^{n-2}}}$

$$\pm \frac{n-5}{(n-2)a^2}\int \frac{x^2}{\sqrt{(x^2 \pm a^2)^{n-2}}}\mathrm{d}x \quad (n \neq 2)$$

354. $\int \frac{x^m}{\sqrt[p]{(x^2 \pm a^2)^n}}\mathrm{d}x = \pm \frac{px^{m+1}}{2a^2(n-p)\sqrt[p]{(x^2 \pm a^2)^{n-p}}}$

$$\pm \frac{2n-p(m+3)}{2a^2(n-p)}\int \frac{x^m}{\sqrt[p]{(x^2 \pm a^2)^{n-p}}}\mathrm{d}x \quad (n \neq p)$$

355. $\int \frac{\mathrm{d}x}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$

356. $\int \frac{\mathrm{d}x}{x\sqrt{x^2-a^2}} = \frac{1}{|a|}\operatorname{arcsec}\frac{x}{a}$

357. $\int \frac{\mathrm{d}x}{x\sqrt{(x^2+a^2)^3}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$

358. $\int \frac{\mathrm{d}x}{x\sqrt{(x^2-a^2)^3}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{|a^3|}\operatorname{arcsec}\frac{x}{a}$

359. $\int \frac{\mathrm{d}x}{x\sqrt{(x^2 \pm a^2)^n}} = \pm \frac{1}{a^2(n-2)\sqrt{(x^2 \pm a^2)^{n-2}}} \pm \frac{1}{a^2}\int \frac{\mathrm{d}x}{x\sqrt{(x^2 \pm a^2)^{n-2}}}$

$(n \neq 2)$

360. $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}$

361. $\int \frac{\mathrm{d}x}{x^2\sqrt{(x^2 \pm a^2)^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{x^2 \pm a^2}}{x} + \frac{x}{\sqrt{x^2 \pm a^2}}\right)$

362. $\int \frac{\mathrm{d}x}{x^2\sqrt{(x^2 \pm a^2)^n}} = \pm \frac{1}{a^2(n-2)x\sqrt{(x^2 \pm a^2)^{n-2}}}$

$$\pm \frac{n-1}{a^2(n-2)}\int \frac{\mathrm{d}x}{x^2\sqrt{(x^2 \pm a^2)^{n-2}}} \quad (n \neq 2)$$

363. $\int \frac{\mathrm{d}x}{x^m\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{(m-1)a^2 x^{m-1}} \mp \frac{m-2}{(m-1)a^2}\int \frac{\mathrm{d}x}{x^{m-2}\sqrt{x^2 \pm a^2}}$

364. $\int \frac{\mathrm{d}x}{x^{2m}\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m-1} \frac{(m-1)!\,m!\,(2r)!\,2^{2m-2r-1}}{(r!)^2(2m)!(\mp a^2)^{m-r}x^{2r+1}}$

365. $\int \frac{\mathrm{d}x}{x^{2m+1}\sqrt{x^2 + a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 + a^2}}{a^2}\sum_{r=1}^{m}(-1)^{m-r+1}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}}\right.$

$$\left. + \frac{(-1)^{m+1}}{2^{2m}a^{2m+1}}\ln\left|\frac{\sqrt{x^2 + a^2} + a}{x}\right|\right]$$

366. $\int \frac{\mathrm{d}x}{x^{2m+1}\sqrt{x^2 - a^2}}$

$$= \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 - a^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}|a|^{2m+1}}\operatorname{arcsec}\frac{x}{a}\right]$$

367. $\int \frac{\mathrm{d}x}{x^m\sqrt[p]{(x^2 \pm a^2)^n}}$

$$= \frac{p}{2(n-p)a^2x^{m-1}\sqrt[p]{(x^2 \pm a^2)^{n-p}}} \pm \frac{2n + p(m-3)}{2(n-p)a^2}\int \frac{\mathrm{d}x}{x^m\sqrt[p]{(x^2 \pm a^2)^{n-p}}}$$

$(n \neq p)$

368. $\int \frac{\sqrt{x^2 + a^2}}{x}\mathrm{d}x = \sqrt{x^2 + a^2} - a\ln\left|\frac{a + \sqrt{x^2 + a^2}}{x}\right|$

369. $\int \frac{\sqrt{x^2 - a^2}}{x}\mathrm{d}x = \sqrt{x^2 - a^2} - |a|\operatorname{arcsec}\frac{x}{a}$

370. $\int \frac{\sqrt{(x^2 + a^2)^3}}{x}\mathrm{d}x = \frac{1}{3}\sqrt{(x^2 + a^2)^3} + a^2\sqrt{x^2 + a^2} - a^3\ln\left|\frac{a + \sqrt{x^2 + a^2}}{x}\right|$

371. $\int \frac{\sqrt{(x^2 - a^2)^3}}{x}\mathrm{d}x = \frac{1}{3}\sqrt{(x^2 - a^2)^3} - a^2\sqrt{x^2 - a^2} + a^3\operatorname{arcsec}\frac{x}{a}$

372. $\int \frac{\sqrt{(x^2 \pm a^2)^n}}{x}\mathrm{d}x = \frac{1}{n}\sqrt{(x^2 \pm a^2)^n} \pm a^2\int \frac{\sqrt{(x^2 \pm a^2)^{n-2}}}{x}\mathrm{d}x$

373. $\int \frac{\sqrt{x^2 \pm a^2}}{x^2}\mathrm{d}x = -\frac{\sqrt{x^2 \pm a^2}}{x} + \ln|x + \sqrt{x^2 \pm a^2}|$

374. $\int \frac{\sqrt{(x^2 \pm a^2)^3}}{x^2}\mathrm{d}x$

$$= -\frac{1}{x}\sqrt{(x^2 \pm a^2)^3} + \frac{3x}{2}\sqrt{x^2 \pm a^2} \pm \frac{3a^2}{2}\ln|x + \sqrt{x^2 \pm a^2}|$$

375. $\int \frac{\sqrt{(x^2 \pm a^2)^n}}{x^2}\mathrm{d}x = \frac{\sqrt{(x^2 \pm a^2)^n}}{(n-1)x} \pm \frac{na^2}{n-1}\int \frac{\sqrt{(x^2 \pm a^2)^{n-2}}}{x^2}\mathrm{d}x \quad (n \neq 1)$

376. $\int \frac{\sqrt[p]{(x^2 \pm a^2)^n}}{x^m}\mathrm{d}x = \frac{p\sqrt[p]{(x^2 \pm a^2)^n}}{(2n - mp + p)x^{m-1}} \pm \frac{2na^2}{2n - mp + p}\int \frac{\sqrt[p]{(x^2 \pm a^2)^{n-p}}}{x^m}\mathrm{d}x$

$[2n \neq (m-1)p]$

377. $\int \frac{x^m}{\sqrt{x^2 \pm a^2}}\mathrm{d}x = \frac{1}{m}x^{m-1}\sqrt{x^2 \pm a^2} \mp \frac{m-1}{m}a^2\int \frac{x^{m-2}}{\sqrt{x^2 \pm a^2}}\mathrm{d}x$

378. $\int \frac{x^{2m}}{\sqrt{x^2 \pm a^2}}\mathrm{d}x = \frac{(2m)!}{2^{2m}(m!)^2}\Big[\sqrt{x^2 \pm a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{(2r)!}(\mp a^2)^{m-r}(2x)^{2r-1}$

$$+ (\mp a^2)^m \ln|x + \sqrt{x^2 \pm a^2}|\Big]$$

379. $\int \frac{x^{2m+1}}{\sqrt{x^2 \pm a^2}}\mathrm{d}x = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(\mp 4a^2)^{m-r}x^{2r}$

380. $\int \frac{\mathrm{d}x}{(x+a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x+a)}$

381. $\int \frac{\mathrm{d}x}{(x-a)\sqrt{x^2 - a^2}} = -\frac{\sqrt{x^2 - a^2}}{a(x-a)}$

Ⅰ.1.2.16 含有 $\sqrt{a^2 - x^2}$ 的积分

382. $\int \sqrt{a^2 - x^2}\mathrm{d}x = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2\arcsin\frac{x}{|a|}\right)$

383. $\int \sqrt{(a^2 - x^2)^3}\mathrm{d}x$

$$= \frac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\arcsin\frac{x}{|a|}\right]$$

384. $\int \sqrt{(a^2 - x^2)^n}\mathrm{d}x = \frac{1}{n+1}\left[x\sqrt{(a^2 - x^2)^n} + na^2\int \sqrt{(a^2 - x^2)^{n-2}}\mathrm{d}x\right]$

$(n \neq -1)$

385. $\int x\sqrt{a^2 - x^2}\mathrm{d}x = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}$

386. $\int x\sqrt{(a^2 - x^2)^3}\mathrm{d}x = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}$

387. $\int x\sqrt{(a^2-x^2)^n}\,\mathrm{d}x=-\frac{1}{n+2}\sqrt{(a^2-x^2)^{n+2}}\quad(n\neq-2)$

388. $\int x^2\sqrt{a^2-x^2}\,\mathrm{d}x=-\frac{x}{4}\sqrt{(a^2-x^2)^3}+\frac{a^2}{8}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{|a|}\right)$

389. $\int x^2\sqrt{(a^2-x^2)^3}\,\mathrm{d}x=-\frac{x}{6}\sqrt{(a^2-x^2)^5}+\frac{a^2x}{24}\sqrt{(a^2-x^2)^3}$
$+\frac{a^4x}{16}\sqrt{a^2-x^2}+\frac{a^6}{16}\arcsin\frac{x}{|a|}$

390. $\int x^2\sqrt{(a^2-x^2)^n}\,\mathrm{d}x$
$=\frac{1}{n+3}\left(x^3\sqrt{(a^2-x^2)^n}+na^2\int x^2\sqrt{(a^2-x^2)^{n-2}}\,\mathrm{d}x\right)\quad(n\neq-3)$
$=-\frac{x}{n+2}\sqrt{(a^2-x^2)^{n+2}}+\frac{1}{n+2}\int\sqrt{(a^2-x^2)^{n+2}}\,\mathrm{d}x\quad(n\neq-2)$

391. $\int x^m\sqrt[p]{(a^2-x^2)^n}\,\mathrm{d}x$
$=\frac{px^{m+1}\sqrt[p]{(a^2-x^2)^n}}{2n+mp+p}+\frac{2na^2}{2n+mp+p}\int x^m\sqrt[p]{(a^2-x^2)^{n-p}}\,\mathrm{d}x$
$[2n\neq-(m+1)p]$

392. $\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{|a|}=-\arccos\frac{x}{|a|}$

393. $\int\frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^3}}=\frac{x}{a^2\sqrt{a^2-x^2}}$

394. $\int\frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^n}}=\frac{x}{(n-2)a^2\sqrt{(a^2-x^2)^{n-2}}}+\frac{n-3}{(n-2)a^2}\int\frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^{n-2}}}$
$(n\neq2)$

395. $\int\frac{x}{\sqrt{a^2-x^2}}\,\mathrm{d}x=-\sqrt{a^2-x^2}$

396. $\int\frac{x}{\sqrt{(a^2-x^2)^3}}\,\mathrm{d}x=\frac{1}{\sqrt{a^2-x^2}}$

397. $\int\frac{x}{\sqrt{(a^2-x^2)^n}}\,\mathrm{d}x=\frac{1}{(n-2)\sqrt{(a^2-x^2)^{n-2}}}\quad(n>2)$

398. $\int\frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{|a|}$

399. $\int\frac{x^2}{\sqrt{(a^2-x^2)^3}}\,\mathrm{d}x=\frac{x}{\sqrt{a^2-x^2}}-\arcsin\frac{x}{|a|}$

400. $\int\frac{x^2}{\sqrt{(a^2-x^2)^n}}\,\mathrm{d}x$
$=\frac{x^3}{(n-2)a^2\sqrt{(a^2-x^2)^{n-2}}}+\frac{n-5}{(n-2)a^2}\int\frac{x^2}{\sqrt{(a^2-x^2)^{n-2}}}\,\mathrm{d}x\quad(n\neq2)$

$$= \frac{x}{(n-2)\sqrt{(a^2-x^2)^{n-2}}} - \frac{1}{n-2}\int \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^{n-2}}} \quad (n \neq 2)$$

401. $\int \frac{x^m}{\sqrt[p]{(a^2-x^2)^n}}\mathrm{d}x$

$$= \frac{px^{m+1}}{2a^2(n-p)\sqrt[p]{(a^2-x^2)^n}} + \frac{2n-(m+3)p}{2a^2(n-p)}\int \frac{x^m}{\sqrt[p]{(a^2-x^2)^{n-p}}}\mathrm{d}x$$

$(n \neq p)$

402. $\int \frac{\mathrm{d}x}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

403. $\int \frac{\mathrm{d}x}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

404. $\int \frac{\mathrm{d}x}{x\sqrt{(a^2-x^2)^n}} = \frac{1}{(n-2)a^2\sqrt{(a^2-x^2)^{n-2}}} + \frac{1}{a^2}\int \frac{\mathrm{d}x}{x\sqrt{(a^2-x^2)^{n-2}}}$

$(n \neq 2)$

405. $\int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$

406. $\int \frac{\mathrm{d}x}{x^2\sqrt{(a^2-x^2)^3}} = \frac{1}{a^4}\left(\frac{x}{\sqrt{a^2-x^2}} - \frac{\sqrt{a^2-x^2}}{x}\right)$

407. $\int \frac{\mathrm{d}x}{x^2\sqrt{(a^2-x^2)^n}}$

$$= \frac{1}{(n-2)a^2x\sqrt{(a^2-x^2)^{n-2}}} + \frac{n-1}{(n-2)a^2}\int \frac{\mathrm{d}x}{x^2\sqrt{(a^2-x^2)^{n-2}}} \quad (n \neq 2)$$

408. $\int \frac{\mathrm{d}x}{x^m\sqrt[p]{(a^2-x^2)^n}}$

$$= \frac{p}{2(n-p)a^2x^{m-1}\sqrt[p]{(a^2-x^2)^{n-p}}} + \frac{2n+(m-3)p}{2(n-p)a^2}\int \frac{\mathrm{d}x}{x^m\sqrt[p]{(a^2-x^2)^{n-p}}}$$

$(n \neq p)$

409. $\int \frac{\sqrt{a^2-x^2}}{x}\mathrm{d}x = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

410. $\int \frac{\sqrt{(a^2-x^2)^3}}{x}\mathrm{d}x = \frac{1}{3}\sqrt{(a^2-x^2)^3} + a^2\sqrt{a^2-x^2} - a^3\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

411. $\int \frac{\sqrt{(a^2-x^2)^n}}{x}\mathrm{d}x = \frac{1}{n}\sqrt{(a^2-x^2)^n} + a^2\int \frac{\sqrt{(a^2-x^2)^{n-2}}}{x}\mathrm{d}x$

412. $\int \frac{\sqrt{a^2-x^2}}{x^2}\mathrm{d}x = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{|a|}$

413. $\int \frac{\sqrt{(a^2-x^2)^3}}{x^2}\mathrm{d}x = -\frac{1}{x}\sqrt{(a^2-x^2)^3} - \frac{3x}{2}\sqrt{a^2-x^2} - \frac{3a^2}{2}\arcsin\frac{x}{a}$

414. $\int \frac{\sqrt{(a^2-x^2)^n}}{x^2}\mathrm{d}x = \frac{1}{(n-1)x}\sqrt{(a^2-x^2)^n} + \frac{na^2}{n-1}\int \frac{\sqrt{(a^2-x^2)^{n-2}}}{x^2}\mathrm{d}x$

$(n \neq 1)$

415. $\int \frac{\sqrt[p]{(a^2-x^2)^n}}{x^m}\mathrm{d}x = \frac{p\sqrt[p]{(a^2-x^2)^n}}{(2n-mp+p)x^{m-1}} + \frac{2na^2}{2n-mp+p}\int \frac{\sqrt[p]{(a^2-x^2)^{n-p}}}{x^m}\mathrm{d}x$

416. $\int \frac{x^m}{\sqrt{a^2-x^2}}\mathrm{d}x = -\frac{x^{m-1}\sqrt{a^2-x^2}}{m} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}}{\sqrt{a^2-x^2}}\mathrm{d}x$

417. $\int \frac{x^{2m}}{\sqrt{a^2-x^2}}\mathrm{d}x$

$$= \frac{(2m)!}{(m!)^2}\left[-\sqrt{a^2-x^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2^{2m-2r+1}(2r)!}a^{2m+2r}x^{2r-1} + \frac{a^{2m}}{2^{2m}}\arcsin\frac{x}{|a|}\right]$$

418. $\int \frac{x^{2m+1}}{\sqrt{a^2-x^2}}\mathrm{d}x = -\sqrt{a^2-x^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(4a^2)^{m-r}x^{2r}$

419. $\int \frac{\mathrm{d}x}{x^m\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{(m-1)a^2x^{m-1}} + \frac{m-2}{(m-1)a^2}\int \frac{\mathrm{d}x}{x^{m-2}\sqrt{a^2-x^2}}$

420. $\int \frac{\mathrm{d}x}{x^{2m}\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\sum_{r=0}^{m-1}\frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!a^{2m-2r}x^{2r+1}}$

421. $\int \frac{\mathrm{d}x}{x^{2m+1}\sqrt{a^2-x^2}} = \frac{(2m)!}{(m!)^2}\left[-\frac{\sqrt{a^2-x^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}}\right.$

$$\left. + \frac{1}{2^{2m}a^{2m+1}}\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|\right]$$

422. $\int \frac{\mathrm{d}x}{(b^2-x^2)\sqrt{a^2-x^2}}$

$$= \begin{cases} \frac{1}{2b\sqrt{a^2-x^2}}\ln\frac{(b\sqrt{a^2-x^2}+x\sqrt{a^2-b^2})^2}{|b^2-x^2|} & (a^2>b^2) \\ \frac{1}{b\sqrt{b^2-a^2}}\arctan\frac{x\sqrt{b^2-a^2}}{b\sqrt{a^2-x^2}} & (a^2<b^2) \end{cases}$$

423. $\int \frac{\mathrm{d}x}{(b^2+x^2)\sqrt{a^2-x^2}} = \frac{1}{b\sqrt{a^2+b^2}}\arctan\frac{x\sqrt{a^2+b^2}}{b\sqrt{a^2-x^2}}$

424. $\int \frac{\sqrt{a^2-x^2}}{b^2+x^2}\mathrm{d}x = \frac{\sqrt{a^2+b^2}}{|b|}\arcsin\frac{x\sqrt{a^2+b^2}}{|a|\sqrt{x^2+b^2}} - \arcsin\frac{x}{|a|}$

Ⅰ.1.2.17 含有 $\sqrt{a+bx+cx^2}$ 的积分

设 $X = a+bx+cx^2$，$q = 4ac-b^2$ 和 $k = \frac{4c}{q}$，$q \neq 0$.（如果 $q = 0$，那么

$\sqrt{X}=\sqrt{c}\left|x+\frac{b}{2c}\right|$，这是 Ⅰ.1.2.1 小节的情形，应该使用其中相应的公式.)

425. $$\int\frac{\mathrm{d}x}{\sqrt{X}}=\begin{cases}\frac{1}{\sqrt{c}}\ln\left|2\sqrt{cX}+2cx+b\right| & (c>0)\\ \frac{1}{\sqrt{c}}\operatorname{arsinh}\frac{2cx+b}{\sqrt{q}} & (c>0,\ q>0)\\ -\frac{1}{\sqrt{-c}}\arcsin\frac{2cx+b}{\sqrt{-q}} & (c<0,\ q<0)\\ \frac{1}{\sqrt{c}}\ln\left|2cx+b\right| & (c>0,\ q=0)\end{cases}$$

426. $$\int\frac{\mathrm{d}x}{X\sqrt{X}}=\frac{2(2cx+b)}{q\sqrt{X}}$$

427. $$\int\frac{\mathrm{d}x}{X^2\sqrt{X}}=\frac{2(2cx+b)}{3q\sqrt{X}}\left(\frac{1}{X}+2k\right)$$

428. $$\int\frac{\mathrm{d}x}{X^n\sqrt{X}}=\frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n}+\frac{2k(n-1)}{2n-1}\int\frac{\mathrm{d}x}{X^{n-1}\sqrt{X}}$$
$$=\frac{(2cx+b)(n!)(n-1)!4^nk^{n-1}}{q(2n)!\sqrt{X}}\sum_{r=0}^{n-1}\frac{(2r)!}{(4kX)^r(r!)^2}$$

429. $$\int\sqrt{X}\mathrm{d}x=\frac{(2cx+b)\sqrt{X}}{4c}+\frac{1}{2k}\int\frac{\mathrm{d}x}{\sqrt{X}}$$

430. $$\int X\sqrt{X}\mathrm{d}x=\frac{(2cx+b)\sqrt{X}}{8c}\left(X+\frac{3}{2k}\right)+\frac{3}{8k^2}\int\frac{\mathrm{d}x}{\sqrt{X}}$$

431. $$\int X^2\sqrt{X}\mathrm{d}x=\frac{(2cx+b)\sqrt{X}}{12c}\left(X^2+\frac{5X}{4k}+\frac{15}{8k^2}\right)+\frac{5}{16k^3}\int\frac{\mathrm{d}x}{\sqrt{X}}$$

432. $$\int X^n\sqrt{X}\mathrm{d}x=\frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c}+\frac{2n+1}{2(n+1)k}\int X^{n-1}\sqrt{X}\mathrm{d}x$$
$$=\frac{(2n+2)!}{[(n+1)!]^2(4k)^{n+1}}\left[\frac{k(2cx+b)\sqrt{X}}{c}\sum_{r=0}^{n}\frac{r!(r+1)!(4kX)^r}{(2r+2)!}+\int\frac{\mathrm{d}x}{\sqrt{X}}\right]$$

433. $$\int x\sqrt{X}\mathrm{d}x=\frac{X\sqrt{X}}{3c}-\frac{b(2cx+b)}{8c^2}\sqrt{X}-\frac{b}{4ck}\int\frac{\mathrm{d}x}{\sqrt{X}}$$

434. $$\int x^2\sqrt{X}\mathrm{d}x=\left(x-\frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c}+\frac{5b^2-4ac}{16c^2}\int\sqrt{X}\mathrm{d}x$$

435. $$\int x^m\sqrt{X}\mathrm{d}x=\frac{x^{m-1}X\sqrt{X}}{(m+2)c}-\frac{(2m+1)b}{2(m+2)c}\int x^{m-1}\sqrt{X}\mathrm{d}x-\frac{(m-1)a}{(m+2)c}\int x^{m-2}\sqrt{X}\mathrm{d}x$$

436. $$\int x^m\sqrt{X^n}\mathrm{d}x=\frac{1}{(m+n+1)c}\left[x^{m-1}\sqrt{X^{n+2}}-\frac{(2m+n)b}{2}\int x^{m-1}\sqrt{X^n}\mathrm{d}x\right.$$
$$\left.-(m-1)a\int x^{m-2}\sqrt{X^n}\mathrm{d}x\right]\quad(n>0)$$

437. $\int \frac{x}{\sqrt{X}}\mathrm{d}x = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{\mathrm{d}x}{\sqrt{X}}$

438. $\int \frac{x^2}{\sqrt{X}}\mathrm{d}x = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2 - 4ac}{8c^2}\int \frac{\mathrm{d}x}{\sqrt{X}}$

439. $\int \frac{x^m}{\sqrt{X}}\mathrm{d}x = \frac{1}{mc}x^{m-1}\sqrt{X} - \frac{(2m-1)b}{2mc}\int \frac{x^{m-1}}{\sqrt{X}}\mathrm{d}x - \frac{(m-1)a}{mc}\int \frac{x^{m-2}}{\sqrt{X}}\mathrm{d}x$

440. $\int \frac{x^m}{\sqrt{X^n}}\mathrm{d}x$

$$= \frac{1}{(m-n+1)c}\left[\frac{x^{m-1}}{\sqrt{X^{n-2}}} - \frac{(2m-n)b}{2}\int \frac{x^{m-1}}{\sqrt{X^n}}\mathrm{d}x - (m-1)a\int \frac{x^{m-2}}{\sqrt{X^n}}\mathrm{d}x\right]$$

$(n > 0)$

441. $\int xX\sqrt{X}\mathrm{d}x = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\mathrm{d}x$

442. $\int xX^n\sqrt{X}\mathrm{d}x = \frac{X^{n+1}\sqrt{X}}{(2n+3)c} - \frac{b}{2c}\int X^n\sqrt{X}\mathrm{d}x$

443. $\int \frac{x}{X\sqrt{X}}\mathrm{d}x = -\frac{2(bx+2a)}{q\sqrt{X}}$

444. $\int \frac{x}{X^n\sqrt{X}}\mathrm{d}x = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{\mathrm{d}x}{X^n\sqrt{X}}$

445. $\int \frac{x^2}{X\sqrt{X}}\mathrm{d}x = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{\mathrm{d}x}{\sqrt{X}}$

446. $\int \frac{x^2}{X^n\sqrt{X}}\mathrm{d}x = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)cq}\int \frac{\mathrm{d}x}{X^{n-1}\sqrt{X}}$

447. $$\int \frac{\mathrm{d}x}{x\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{-a}}\arcsin\frac{bx+2a}{|x|\sqrt{-q}} & (a<0) \\ -\frac{2\sqrt{X}}{bx} & (a=0) \\ -\frac{1}{\sqrt{a}}\ln\left|\frac{2\sqrt{aX}+bx+2a}{x}\right| & (a>0) \end{cases}$$

448. $\int \frac{\mathrm{d}x}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{\mathrm{d}x}{x\sqrt{X}}$

449. $\int \frac{\mathrm{d}x}{x^m\sqrt{X}} = -\frac{\sqrt{X}}{(m-1)ax^{m-1}} - \frac{(2m-3)b}{2(m-1)a}\int \frac{\mathrm{d}x}{x^{m-1}\sqrt{X}} - \frac{(m-2)c}{(m-1)a}\int \frac{\mathrm{d}x}{x^{m-2}\sqrt{X}}$

$(m \neq 1)$

450. $\int \frac{\mathrm{d}x}{x\sqrt{X^n}} = \frac{2}{(n-2)a\sqrt{X^{n-2}}} - \frac{b}{2a}\int \frac{\mathrm{d}x}{\sqrt{X^n}} + \frac{1}{a}\int \frac{\mathrm{d}x}{x\sqrt{X^{n-2}}} \quad (n \neq 2)$

451. $\int \frac{\mathrm{d}x}{x^m\sqrt{X^n}} = -\frac{1}{(m-1)ax^{m-1}\sqrt{X^{n-2}}} - \frac{(n+2m-4)b}{2(m-1)a}\int \frac{\mathrm{d}x}{x^{m-1}\sqrt{X^n}}$

$$-\frac{(n+m-3)c}{(m-1)a}\int\frac{\mathrm{d}x}{x^{m-2}\sqrt{X^n}}\quad(m\neq 1)$$

452. $\int\frac{\sqrt{X}}{x}\mathrm{d}x=\sqrt{X}+\frac{b}{2}\int\frac{\mathrm{d}x}{\sqrt{X}}+a\int\frac{\mathrm{d}x}{x\sqrt{X}}$

453. $\int\frac{\sqrt{X}}{x^2}\mathrm{d}x=-\frac{\sqrt{X}}{x}+\frac{b}{2}\int\frac{\mathrm{d}x}{x\sqrt{X}}+c\int\frac{\mathrm{d}x}{\sqrt{X}}$

454. $\int\frac{\sqrt{X}}{x^m}\mathrm{d}x=-\frac{\sqrt{X}}{(m-1)x^{m-1}}+\frac{b}{2(m-1)}\int\frac{\mathrm{d}x}{x^{m-1}\sqrt{X}}+\frac{c}{m-1}\int\frac{\mathrm{d}x}{x^{m-2}\sqrt{X}}$

$(m\neq 1)$

455. $\int\frac{\sqrt{X^n}}{x^m}\mathrm{d}x$

$$=-\frac{\sqrt{X^{n+2}}}{(m-1)ax^{m-1}}+\frac{(n-2m+4)b}{2(m-1)a}\int\frac{\sqrt{X^n}}{x^{m-1}}\mathrm{d}x+\frac{(n-m+3)c}{(m-1)a}\int\frac{\sqrt{X^n}}{x^{m-2}}\mathrm{d}x$$

$(m\neq 1)$

456. $\int\frac{e+fx}{\sqrt{X}}\mathrm{d}x=\frac{f}{c}\sqrt{X}+\frac{2ce-bf}{2c}\int\frac{\mathrm{d}x}{\sqrt{X}}$

457. $\int\frac{b+2cx}{\sqrt{X}}\mathrm{d}x=2\sqrt{X}$

Ⅰ.1.2.18 含有 $\sqrt{bx+cx^2}$ 和 $\sqrt{bx-cx^2}$ 的积分

458. $\int\sqrt{bx+cx^2}\mathrm{d}x=\frac{b+2cx}{4c}\sqrt{bx+cx^2}-\frac{b^2}{8\sqrt{c^3}}\operatorname{arcosh}\frac{b+2cx}{b}$

459. $\int\sqrt{bx-cx^2}\mathrm{d}x=\frac{2cx-b}{4c}\sqrt{bx-cx^2}+\frac{b^2}{4\sqrt{c^3}}\arcsin\sqrt{\frac{cx}{b}}$

460. $\int x\sqrt{bx+cx^2}\mathrm{d}x$

$$=\frac{1}{3c}\sqrt{(bx+cx^2)^3}-\frac{b(b+2cx)}{8c^2}\sqrt{bx+cx^2}+\frac{b^2}{16c^2}\int\frac{\mathrm{d}x}{\sqrt{bx+cx^2}}$$

461. $\int x^m\sqrt{bx\pm cx^2}\mathrm{d}x$

$$=\pm\frac{x^{m-1}}{(m+2)c}\sqrt{(bx\pm cx^2)^3}\mp\frac{(2m+1)b}{2(m+2)c}\int x^{m-1}\sqrt{bx\pm cx^2}\mathrm{d}x$$

462. $\int\frac{\sqrt{bx+cx^2}}{x}\mathrm{d}x=\sqrt{bx+cx^2}+\frac{b}{2\sqrt{c}}\operatorname{arcosh}\frac{b+2cx}{b}$

463. $\int\frac{\sqrt{bx-cx^2}}{x}\mathrm{d}x=\sqrt{bx-cx^2}+\frac{b}{\sqrt{c}}\arcsin\sqrt{\frac{cx}{b}}$

464. $\int \frac{\sqrt{bx+cx^2}}{x^2}\mathrm{d}x = -\frac{2\sqrt{bx+cx^2}}{x} + c\int \frac{\mathrm{d}x}{\sqrt{bx+cx^2}}$

465. $\int \frac{\sqrt{bx\pm cx^2}}{x^m}\mathrm{d}x = \pm\frac{2\sqrt{(bx\pm cx^2)^3}}{(2m-3)bx^{m+1}} \mp \frac{2(m-3)c}{(2m-3)b}\int \frac{\sqrt{bx\pm cx^2}}{x^{m-1}}\mathrm{d}x$

466. $\int \frac{\sqrt{(bx+cx^2)^3}}{x}\mathrm{d}x = \frac{2\sqrt{(bx+cx^2)^5}}{3bx} - \frac{8c}{3b}\int \sqrt{(bx+cx^2)^3}\mathrm{d}x$

467. $\int \frac{\sqrt{(bx+cx^2)^3}}{x^2}\mathrm{d}x = \frac{\sqrt{(bx+cx^2)^3}}{2x} + \frac{3b\sqrt{bx+cx^2}}{4} + \frac{3b^2}{8}\int \frac{\mathrm{d}x}{\sqrt{bx+cx^2}}$

468. $\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m}\mathrm{d}x$

$$= \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b}\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}}\mathrm{d}x$$

469. $\int \frac{\mathrm{d}x}{\sqrt{bx+cx^2}} = \frac{1}{\sqrt{c}}\operatorname{arcosh}\frac{b+2cx}{b}$

470. $\int \frac{\mathrm{d}x}{\sqrt{bx-cx^2}} = \frac{2}{\sqrt{c}}\arcsin\sqrt{\frac{cx}{b}}$

471. $\int \frac{\mathrm{d}x}{x\sqrt{bx\pm cx^2}} = \pm\frac{2}{bx}\sqrt{bx\pm cx^2}$

472. $\int \frac{\mathrm{d}x}{x^2\sqrt{bx+cx^2}} = \frac{2}{3}\left(-\frac{1}{bx^2} + \frac{2c}{b^2x}\right)\sqrt{bx+cx^2}$

473. $\int \frac{\mathrm{d}x}{x\sqrt{(bx+cx^2)^3}} = \frac{2}{3}\left(-\frac{1}{bx} + \frac{4c}{b^2} + \frac{8c^2x}{b^3}\right)\frac{1}{\sqrt{bx+cx^2}}$

474. $\int \frac{\mathrm{d}x}{x^2\sqrt{(bx+cx^2)^3}} = \frac{2}{5}\left(-\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^3x}{b^4}\right)\frac{1}{\sqrt{bx+cx^2}}$

475. $\int \frac{x^m}{\sqrt{bx\pm cx^2}}\mathrm{d}x = \pm\frac{x^{m-1}}{mc}\sqrt{bx\pm cx^2} \mp \frac{(2m\mp 1)b}{2mc}\int \frac{x^{m-1}}{\sqrt{bx\pm cx^2}}\mathrm{d}x$

476. $\int \frac{\mathrm{d}x}{x^m\sqrt{bx\pm cx^2}} = -\frac{2\sqrt{bx\pm cx^2}}{(2m-1)bx^m} \mp \frac{2(m-1)c}{(2m-1)b}\int \frac{\mathrm{d}x}{x^{m-1}\sqrt{bx\pm cx^2}}$

477. $\int \frac{\mathrm{d}x}{x^m\sqrt{(bx+cx^2)^{2n+1}}} = -\frac{2}{(2m+2n-1)bx^m\sqrt{(bx+cx^2)^{2n-1}}}$

$$-\frac{(2m+4n-2)c}{(2m+2n-1)b}\int \frac{\mathrm{d}x}{x^{m-1}\sqrt{(bx+cx^2)^{2n+1}}}$$

Ⅰ.1.2.19 含有 $\sqrt{a+cx^2}$ 和 x^n 的积分

设 $u = \sqrt{a+cx^2}$,并令

$$I_1 = \begin{cases} \dfrac{1}{\sqrt{c}}\ln(x\sqrt{c} + u) & (c > 0) \\ \dfrac{1}{\sqrt{-c}}\arcsin\left(x\sqrt{-\dfrac{c}{a}}\right) & (c < 0,\ a > 0) \end{cases}$$

$$I_2 = \begin{cases} \dfrac{1}{2\sqrt{a}}\ln\dfrac{u - \sqrt{a}}{u + \sqrt{a}} & (a > 0,\ c > 0) \\ \dfrac{1}{2\sqrt{a}}\ln\dfrac{\sqrt{a} - u}{\sqrt{a} + u} & (a > 0,\ c < 0) \\ \dfrac{1}{\sqrt{-a}}\arccos\left(\dfrac{1}{x}\sqrt{-\dfrac{a}{c}}\right) & (a < 0,\ c > 0) \end{cases}$$

478. $\int u\,\mathrm{d}x = \frac{1}{2}xu + \frac{1}{2}aI_1$

479. $\int u^3\,\mathrm{d}x = \frac{1}{4}xu^3 + \frac{3}{8}axu + \frac{3}{8}a^2 I_1$

480. $\int u^{2n+1}\,\mathrm{d}x = \frac{xu^{2n+1}}{2(n+1)} + \frac{(2n+1)a}{2(n+1)}\int u^{2n-1}\,\mathrm{d}x$

481. $\int \frac{\mathrm{d}x}{u} = I_1$

482. $\int \frac{\mathrm{d}x}{u^3} = \frac{1}{a}\frac{x}{u}$

483. $\int \frac{\mathrm{d}x}{u^{2n+1}} = \frac{1}{a^n}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}\binom{n-1}{k}\frac{c^k x^{2k+1}}{u^{2k+1}}$

484. $\int xu\,\mathrm{d}x = \frac{1}{3c}u^3$

485. $\int xu^3\,\mathrm{d}x = \frac{1}{5c}u^5$

486. $\int xu^{2n+1}\,\mathrm{d}x = \frac{1}{(2n+3)c}u^{2n+3}$

487. $\int x^2 u\,\mathrm{d}x = \frac{1}{4}\frac{xu^3}{c} - \frac{1}{8}\frac{axu}{c} - \frac{1}{8}\frac{a^2}{c}I_1$

488. $\int x^2 u^3\,\mathrm{d}x = \frac{1}{6}\frac{xu^5}{c} - \frac{1}{24}\frac{axu^3}{c} - \frac{1}{16}\frac{a^2 xu}{c} - \frac{1}{16}\frac{a^3}{c}I_1$

489. $\int x^m u^{2n+1}\,\mathrm{d}x = \frac{x^{m-1}u^{2n+3}}{(2n+m+2)c} - \frac{(m-1)a}{(2n+m+2)c}\int x^{m-2}u^{2n+1}\,\mathrm{d}x$

$$= \frac{x^{m+1}u^{2n+1}}{2n+m+2} + \frac{(2n+1)a}{2n+m+2}\int x^m u^{2n-1}\,\mathrm{d}x$$

490. $\int \frac{x}{u}\,\mathrm{d}x = \frac{u}{c}$

491. $\int \frac{x}{u^3}\,\mathrm{d}x = -\frac{1}{cu}$

492. $\int \frac{x}{u^{2n+1}}\mathrm{d}x = -\frac{1}{(2n-1)cu^{2n-1}}$

493. $\int \frac{x^2}{u}\mathrm{d}x = \frac{1}{2}\frac{xu}{c} - \frac{1}{2}\frac{a}{c}I_1$

494. $\int \frac{x^2}{u^3}\mathrm{d}x = -\frac{x}{cu} + \frac{1}{c}I_1$

495. $\int \frac{x^2}{u^{2n+1}}\mathrm{d}x = \frac{1}{a^{n-1}}\sum_{k=0}^{n-2}\frac{(-1)^k}{2k+3}\binom{n-2}{k}\frac{c^k x^{2k+3}}{u^{2k+3}}$

496. $\int \frac{x^3}{u^{2n+1}}\mathrm{d}x = -\frac{1}{(2n-3)c^2u^{2n-3}} + \frac{a}{(2n-1)c^2u^{2n-1}}$

497. $\int \frac{x^m}{u^{2n+1}}\mathrm{d}x = \frac{x^{m+1}}{(2n-1)au^{2n-1}} + \frac{2n-m-2}{(2n-1)a}\int \frac{x^m}{u^{2n-1}}\mathrm{d}x$

$= \frac{x^{m-1}}{(m-2n)cu^{2n-1}} - \frac{(m-1)a}{(m-2n)c}\int \frac{x^{m-2}}{u^{2n+1}}\mathrm{d}x \quad (m \neq 2n)$

498. $\int \frac{u}{x}\mathrm{d}x = u + aI_2$

499. $\int \frac{u^3}{x}\mathrm{d}x = \frac{1}{3}u^3 + au + a^2I_2$

500. $\int \frac{u^{2n+1}}{x}\mathrm{d}x = \frac{u^{2n+1}}{2n+1} + a\int \frac{u^{2n-1}}{x}\mathrm{d}x$

501. $\int \frac{u}{x^2}\mathrm{d}x = -\frac{u}{x} + cI_1$

502. $\int \frac{u^3}{x^2}\mathrm{d}x = -\frac{u^3}{x} + \frac{3}{2}cxu + \frac{3}{2}aI_1$

503. $\int \frac{u^{2n+1}}{x^m}\mathrm{d}x = -\frac{u^{2n+3}}{(m-1)ax^{m-1}} + \frac{(2n-m+4)c}{(m-1)a}\int \frac{u^{2n+1}}{x^{m-2}}\mathrm{d}x \quad (m \neq 1)$

$= \frac{u^{2n+1}}{(2n-m+2)x^{m-1}} + \frac{(2n+1)a}{2n-m+2}\int \frac{u^{2n-1}}{x^m}\mathrm{d}x$

504. $\int \frac{\mathrm{d}x}{xu} = I_2$

505. $\int \frac{\mathrm{d}x}{xu^3} = \frac{1}{a}\left(I_2 + \frac{1}{u}\right)$

506. $\int \frac{\mathrm{d}x}{xu^{2n+1}} = \frac{1}{a^n}I_2 + \sum_{k=0}^{n-1}\frac{1}{(2k+1)a^{n-k}u^{2k+1}}$

507. $\int \frac{\mathrm{d}x}{x^2u^{2n+1}} = -\frac{1}{a^{n+1}}\left[\frac{u}{x} + \sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k-1}\binom{n}{k}c^k\left(\frac{x}{u}\right)^{2k-1}\right]$

508. $\int \frac{\mathrm{d}x}{x^mu} = -\frac{u}{(m-1)ax^{m-1}} - \frac{(m-2)c}{(m-1)a}\int \frac{\mathrm{d}x}{x^{m-2}u} \quad (m \neq 1)$

509. $\int \frac{\mathrm{d}x}{x^mu^{2n+1}} = \frac{1}{(2n-1)ax^{m-1}u^{2n-1}} + \frac{2n+m-2}{(2n-1)a}\int \frac{\mathrm{d}x}{x^mu^{2n-1}}$

$$= -\frac{1}{(m-1)ax^{m-1}u^{2n-1}} - \frac{(2n+m-2)c}{(m-1)a}\int\frac{\mathrm{d}x}{x^{m-2}u^{2n+1}} \quad (m \neq 1)$$

Ⅰ.1.2.20 含有 $\sqrt{2ax-x^2}$ 和 $\sqrt{2ax+x^2}$ 的积分

510. $\int\sqrt{2ax-x^2}\mathrm{d}x = \frac{1}{2}\left[(x-a)\sqrt{2ax-x^2} + a^2\arcsin\frac{x-a}{|a|}\right]$

511. $\int x^n\sqrt{2ax-x^2}\mathrm{d}x$

$$= -\frac{x^{n-1}\sqrt{(2ax-x^2)^3}}{n+2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{2ax-x^2}\mathrm{d}x$$

$$= \sqrt{2ax-x^2}\left[\frac{x^{n+1}}{n+2} - \sum_{r=0}^{n}\frac{(2n+1)!(r!)^2a^{n-r+1}}{2^{n-r}(2r+1)!(n+2)!n!}x^r\right]$$

$$+ \frac{(2n+1)!a^{n+2}}{2^n n!(n+2)!}\arcsin\frac{x-a}{|a|}$$

512. $\int\frac{\sqrt{2ax-x^2}}{x^n}\mathrm{d}x = \frac{\sqrt{(2ax-x^2)^3}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a}\int\frac{\sqrt{2ax-x^2}}{x^{n-1}}\mathrm{d}x$

513. $\int\frac{\mathrm{d}x}{\sqrt{2ax-x^2}} = \arccos\frac{a-x}{|a|} = \arcsin\frac{x-a}{|a|}$

514. $\int\frac{\mathrm{d}x}{\sqrt{(2ax-x^2)^3}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}$

515. $\int\frac{x}{\sqrt{(2ax-x^2)^3}}\mathrm{d}x = \frac{x}{a\sqrt{2ax-x^2}}$

516. $\int\frac{x^n}{\sqrt{2ax-x^2}}\mathrm{d}x$

$$= -\frac{x^{n-1}\sqrt{2ax-x^2}}{n} + \frac{a(2n-1)}{n}\int\frac{x^{n-1}}{\sqrt{2ax-x^2}}\mathrm{d}x$$

$$= -\sqrt{2ax-x^2}\sum_{r=1}^{n}\frac{(2n)!r!(r-1)!a^{n-r}}{2^{n-r}(2r)!(n!)^2}x^{r-1} + \frac{(2n)!a^n}{2^n(n!)^2}\arcsin\frac{x-a}{|a|}$$

517. $\int\frac{\mathrm{d}x}{x^n\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{a(1-2n)x^n} + \frac{n-1}{(2n-1)a}\int\frac{\mathrm{d}x}{x^{n-1}\sqrt{2ax-x^2}}$

$$= -\sqrt{2ax-x^2}\sum_{r=0}^{n-1}\frac{2^{n-r}(n-1)!n!(2r)!}{(2n)!(r!)^2a^{n-r}x^{r+1}}$$

518. $\int\frac{\mathrm{d}x}{\sqrt{2ax+x^2}} = \ln|x+a+\sqrt{2ax+x^2}|$

Ⅰ.1.2.21 其他形式的代数函数的积分

519. $$\int \frac{\mathrm{d}x}{x\sqrt{ax^n+c}} = \begin{cases} \dfrac{1}{n\sqrt{c}}\ln\left|\dfrac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}}\right| & (c>0) \\ \dfrac{2}{n\sqrt{c}}\ln\left|\dfrac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{x^n}}\right| & (c>0) \\ \dfrac{2}{n\sqrt{-c}}\operatorname{arcsec}\sqrt{-\dfrac{ax^n}{c}} & (c<0) \end{cases}$$

520. $$\int \sqrt{ax^2-bx}\,\mathrm{d}x$$
$$= \frac{2ax-b}{4a}\sqrt{ax^2-bx} - \frac{b^2}{8\sqrt{a^3}}\ln\left|2ax-b+2\sqrt{a}\sqrt{ax^2-bx}\right|$$

521. $$\int \frac{\mathrm{d}x}{\sqrt{ax^2-bx}} = \frac{2}{\sqrt{a}}\ln(\sqrt{ax}+\sqrt{ax-b})$$

522. $$\int \frac{\sqrt{ax^2-bx}}{x}\mathrm{d}x = \sqrt{ax^2-bx} - \frac{b}{\sqrt{a}}\ln(\sqrt{ax}+\sqrt{ax-b})$$

523. $$\int \frac{\mathrm{d}x}{x\sqrt{ax^2-bx}} = \frac{2}{bx}\sqrt{ax^2-bx}$$

524. $$\int x^m\sqrt{ax^2-bx}\,\mathrm{d}x$$
$$= -\frac{x^{m-1}}{(m+2)a}\sqrt{(ax^2-bx)^3} + \frac{(2m+1)b}{2(m+1)a}\int x^{m-1}\sqrt{ax^2-bx}\,\mathrm{d}x$$

525. $$\int \frac{\sqrt{ax^2-bx}}{x^m}\mathrm{d}x = \frac{2\sqrt{(ax^2-bx)^3}}{(2m-3)bx^{m-1}} + \frac{2(m-3)a}{(2m-3)b}\int \frac{\sqrt{ax^2-bx}}{x^{m-1}}\mathrm{d}x$$

526. $$\int \frac{x^m}{\sqrt{ax^2-bx}}\mathrm{d}x = \frac{x^{m-1}}{ma}\sqrt{ax^2-bx} + \frac{(2m-1)b}{2ma}\int \frac{x^{m-1}}{\sqrt{ax^2-bx}}\mathrm{d}x$$

527. $$\int \frac{\mathrm{d}x}{x^m\sqrt{ax^2-bx}} = \frac{2\sqrt{ax^2-bx}}{(2m-1)bx^m} + \frac{2(m-1)a}{(2m-1)b}\int \frac{\mathrm{d}x}{x^{m-1}\sqrt{ax^2-bx}}$$

528. $$\int \frac{\mathrm{d}x}{x\sqrt{x^n+a^2}} = -\frac{2}{na}\ln\frac{a+\sqrt{x^n+a^2}}{\sqrt{x^n}}$$

529. $$\int \frac{\mathrm{d}x}{x\sqrt{x^n-a^2}} = -\frac{2}{na}\arcsin\frac{a}{\sqrt{x^n}}$$

530. $$\int \sqrt{\frac{1+x}{1-x}}\mathrm{d}x = \arcsin x - \sqrt{1-x^2}$$

531. $$\int \sqrt{\frac{x}{a^3-x^3}}\mathrm{d}x = \frac{2}{3}\arcsin\left(\frac{x}{a}\right)^{\frac{3}{2}}$$

532. $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\mathrm{d}x = \frac{1}{\sqrt{2}}\ln\left|\frac{x\sqrt{2}+\sqrt{1+x^4}}{1-x^2}\right|$

533. $\int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\mathrm{d}x = \frac{1}{\sqrt{2}}\arctan\frac{x\sqrt{2}}{\sqrt{1+x^4}}$

534. $\int \sqrt{\frac{a+bx}{a-bx}}\mathrm{d}x = -\frac{1}{b}\sqrt{(a+bx)(a-bx)} + \frac{a}{b}\arcsin\frac{bx}{a}$

535. $\int \sqrt{\frac{a-bx}{a+bx}}\mathrm{d}x = \frac{1}{b}\sqrt{(a+bx)(a-bx)} + \frac{a}{b}\arcsin\frac{bx}{a}$

536. $\int \sqrt[p]{(a+bx^q)^n}\mathrm{d}x = \frac{p}{nq+p}\left[x\sqrt[p]{(a+bx^q)^n} + \frac{nqa}{p}\int\sqrt[p]{(a+bx^q)^{n-p}}\mathrm{d}x\right]$

537. $\int \frac{\mathrm{d}x}{\sqrt[p]{(a+bx^q)^n}}$

$$= \frac{p}{qa(n-p)}\left[\frac{x}{\sqrt[p]{(a+bx^q)^{n-p}}} + \frac{nq-(q+1)p}{p}\int\frac{\mathrm{d}x}{\sqrt[p]{(a+bx^q)^{n-p}}}\right]$$

538. $\int x^m\sqrt[p]{(a+bx^q)^n}\mathrm{d}x$

$$= \frac{p}{(m+1)p+nq}\left[x^{m+1}\sqrt[p]{(a+bx^q)^n} + \frac{nqa}{p}\int x^m\sqrt[p]{(a+bx^q)^{n-p}}\mathrm{d}x\right]$$

539. $\int \sqrt[p]{(ax^q+bx^{q+r})^n}\mathrm{d}x = \frac{nx}{(q+r)n+p}\sqrt[p]{(ax^q+bx^{q+r})^n}$

$$+ \frac{n^2ra}{(q+r)np+p^2}\int x^q\sqrt[p]{(ax^q+bx^{q+r})^{n-p}}\mathrm{d}x$$

540. $\int \frac{\mathrm{d}x}{\sqrt[p]{(ax^q+bx^{q+r})^n}} = -\frac{px}{ra(n+p)}\sqrt[p]{(ax^q+bx^{q+r})^{n+p}}$

$$+ \frac{(1+r)p+(q+r)n}{ra(n+p)}\int\frac{x^q}{\sqrt[p]{(ax^q+bx^{q+r})^{n-p}}}\mathrm{d}x$$

541. $\int x^m\sqrt[p]{(ax^q+bx^{q+r})^n}\mathrm{d}x = \frac{px^{m+1}}{(m+1)p+(q+r)n}\sqrt[p]{(ax^q+bx^{q+r})^n}$

$$+ \frac{nar}{(m+1)p+(q+r)n}\int x^{m+q}\sqrt[p]{(ax^q+bx^{q+r})^{n-p}}\mathrm{d}x$$

542. $\int \frac{x^{p-1}}{x^{2m+1}+a^{2m+1}}\mathrm{d}x$

$$= \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\arctan\frac{x+a\cos\frac{2k\pi}{2m+1}}{a\sin\frac{2k\pi}{2m+1}}$$

$$- \frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2+a^2+2ax\cos\frac{2k\pi}{2m+1}\right)$$

$$+ \frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\ln|x+a| \quad (2m+1\geqslant p>0)$$

543. $\int \frac{x^{p-1}}{x^{2m+1}-a^{2m+1}}\mathrm{d}x$

$$= -\frac{2}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\arctan\frac{x-a\cos\frac{2k\pi}{2m+1}}{a\sin\frac{2k\pi}{2m+1}}$$

$$+\frac{1}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2+a^2-2ax\cos\frac{2k\pi}{2m+1}\right)$$

$$+\frac{1}{(2m+1)a^{2m-p+1}}\ln|x-a| \quad (2m+1\geqslant p>0)$$

544. $\int \frac{x^{p-1}}{x^{2m}+a^{2m}}\mathrm{d}x$

$$= \frac{1}{ma^{2m-p}}\sum_{k=1}^{m}\sin\frac{(2k-1)p\pi}{2m}\arctan\frac{x+a\cos\frac{(2k-1)\pi}{2m}}{a\sin\frac{(2k-1)\pi}{2m}}$$

$$-\frac{1}{2ma^{2m-p}}\sum_{k=1}^{m}\cos\frac{(2k-1)p\pi}{2m}\ln\left(x^2+a^2+2ax\cos\frac{(2k-1)\pi}{2m}\right)$$

$(2m\geqslant p>0)$

545. $\int \frac{x^{p-1}}{x^{2m}-a^{2m}}\mathrm{d}x = -\frac{1}{na^{2m-p}}\sum_{k=1}^{m-1}\sin\frac{kp\pi}{m}\arctan\frac{x-a\cos\frac{k\pi}{m}}{a\sin\frac{k\pi}{m}}$

$$+\frac{1}{2ma^{2m-p}}\sum_{k=1}^{m-1}\cos\frac{kp\pi}{m}\ln\left(x^2+a^2-2ax\cos\frac{k\pi}{m}\right)$$

$$+\frac{1}{2ma^{2m-p}}[\ln|x-a|+(-1)^p\ln|x+a|]$$

$(2m\geqslant p>0)$

546. $\int \frac{x^m}{(x-a_1)(x-a_2)\cdots(x-a_k)}\mathrm{d}x$

$$= \frac{a_1{}^m\ln|x-a_1|}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_k)}+\frac{a_2{}^m\ln|x-a_2|}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_k)}$$

$$+\cdots+\frac{a_k{}^m\ln|x-a_k|}{(a_k-a_1)(a_k-a_2)\cdots(a_k-a_{k-1})} \quad (a_i\neq a_j\neq 0)$$

Ⅰ.1.3 三角函数和反三角函数的不定积分

Ⅰ.1.3.1 含有 $\sin^n ax, \cos^n ax, \tan^n ax, \cot^n ax, \sec^n ax, \csc^n ax$ 的积分

547. $\int \sin ax \mathrm{d}x = -\frac{1}{a}\cos ax$

548. $\int \sin^2 ax \mathrm{d}x = \frac{x}{2} - \frac{1}{2a}\cos ax \sin ax = \frac{x}{2} - \frac{1}{4a}\sin 2ax$

549. $\int \sin^3 ax \mathrm{d}x = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$

550. $\int \sin^4 ax \mathrm{d}x = -\frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a} + \frac{3x}{8}$

551. $\int \sin^n ax \mathrm{d}x = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax \mathrm{d}x$

552. $\int \sin^{2m} ax \mathrm{d}x = -\frac{\cos ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\sin^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x$

553. $\int \sin^{2m+1} ax \mathrm{d}x = -\frac{\cos ax}{a}\sum_{r=0}^{m-1}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\sin^{2r} ax$

554. $\int \cos ax \mathrm{d}x = \frac{1}{a}\sin ax$

555. $\int \cos^2 ax \mathrm{d}x = \frac{\sin 2ax}{4a} + \frac{x}{2}$

556. $\int \cos^3 ax \mathrm{d}x = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$

557. $\int \cos^4 ax \mathrm{d}x = \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a} + \frac{3x}{8}$

558. $\int \cos^n ax \mathrm{d}x = \frac{1}{na}\cos^{n-1} ax \sin ax + \frac{n-1}{n}\int \cos^{n-2} ax \mathrm{d}x$

559. $\int \cos^{2m} ax \mathrm{d}x = \frac{\sin ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\cos^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x$

560. $\int \cos^{2m+1} ax \mathrm{d}x = \frac{\sin ax}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\cos^{2r} ax$

561. $\int \tan ax \mathrm{d}x = -\frac{1}{a}\ln|\cos ax| = \frac{1}{a}\ln|\sec ax|$

562. $\int \tan^2 ax \mathrm{d}x = \frac{1}{a}\tan ax - x$

563. $\int \tan^3 ax\mathrm{d}x = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\ln|\cos ax|$

564. $\int \tan^4 ax\mathrm{d}x = \frac{1}{3a}\tan^3 ax - \frac{1}{a}\tan ax + x$

565. $\int \tan^n ax\mathrm{d}x = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\mathrm{d}x \quad (n \neq 1)$

566. $\int \cot ax\mathrm{d}x = \frac{1}{a}\ln|\sin ax| = -\frac{1}{a}\ln|\csc ax|$

567. $\int \cot^2 ax\mathrm{d}x = -\frac{1}{a}\cot ax - x$

568. $\int \cot^3 ax\mathrm{d}x = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\ln|\sin ax|$

569. $\int \cot^4 ax\mathrm{d}x = -\frac{1}{3a}\cot^3 ax + \frac{1}{a}\cot ax + x$

570. $\int \cot^n ax\mathrm{d}x = -\frac{1}{a(n-1)}\cot^{n-1} ax - \int \cot^{n-2} ax\mathrm{d}x \quad (n \neq 1)$

571. $\int \frac{\mathrm{d}x}{\sin ax} = \int \csc ax\mathrm{d}x = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$

572. $\int \frac{\mathrm{d}x}{\sin^2 ax} = \int \csc^2 ax\mathrm{d}x = -\frac{1}{a}\cot ax$

573. $\int \frac{\mathrm{d}x}{\sin^3 ax} = \int \csc^3 ax\mathrm{d}x = -\frac{\cos ax}{2a\sin^2 ax} - \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right|$

574. $\int \frac{\mathrm{d}x}{\sin^4 ax} = \int \csc^4 ax\mathrm{d}x = -\frac{\cot ax}{a} - \frac{\cot^3 ax}{3a}$

575. $\int \frac{\mathrm{d}x}{\sin^m ax} = \int \csc^m ax\mathrm{d}x = -\frac{1}{a(m-1)}\frac{\cos ax}{\sin^{m-1} ax} + \frac{m-2}{m-1}\int \frac{\mathrm{d}x}{\sin^{m-2} ax}$

576. $\int \frac{\mathrm{d}x}{\sin^{2m} ax} = \int \csc^{2m} ax\mathrm{d}x = -\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\sin^{2r+1} ax}$

577. $\int \frac{\mathrm{d}x}{\sin^{2m+1} ax} = \int \csc^{2m+1} ax\mathrm{d}x$

$$= -\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2\sin^{2r+2} ax} + \frac{1}{a}\frac{(2m)!}{2^{2m}(m!)^2}\ln\left|\tan\frac{ax}{2}\right|$$

578. $\int \frac{\mathrm{d}x}{\cos ax} = \int \sec ax\mathrm{d}x = \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right|$

579. $\int \frac{\mathrm{d}x}{\cos^2 ax} = \int \sec^2 ax\mathrm{d}x = \frac{1}{a}\tan ax$

580. $\int \frac{\mathrm{d}x}{\cos^3 ax} = \int \sec^3 ax\mathrm{d}x = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right|$

581. $\int \frac{\mathrm{d}x}{\cos^4 ax} = \int \sec^4 ax\mathrm{d}x = \frac{\tan ax}{a} + \frac{\tan^3 ax}{3a}$

582. $\int \frac{\mathrm{d}x}{\cos^m ax} = \int \sec^m ax\mathrm{d}x = \frac{1}{a(m-1)}\frac{\sin ax}{\cos^{m-1} ax} + \frac{m-2}{m-1}\int \frac{\mathrm{d}x}{\cos^{m-2} ax}$

583. $\int \frac{\mathrm{d}x}{\cos^{2m}ax} = \int \sec^{2m}ax\mathrm{d}x = \frac{1}{a}\sin ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\cos^{2r+1}ax}$

584. $\int \frac{\mathrm{d}x}{\cos^{2m+1}ax} = \int \sec^{2m+1}ax\mathrm{d}x$

$$= \frac{1}{a}\sin ax\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)!\cos^{2r+2}ax} + \frac{1}{a}\frac{(2m)!}{2^{2m}(m!)^2}\ln|\sec ax+\tan ax|$$

Ⅰ.1.3.2 含有 $\sin^m ax\cos^n ax$ 的积分

585. $\int \sin ax\cos ax\mathrm{d}x = \frac{1}{2a}\sin^2 ax$

586. $\int \sin^2 ax\cos ax\mathrm{d}x = \frac{\sin^3 ax}{3a}$

587. $\int \sin^m ax\cos ax\mathrm{d}x = \frac{\sin^{m+1}ax}{(m+1)a} \quad (m\neq -1)$

588. $\int \sin ax\cos^2 ax\mathrm{d}x = -\frac{\cos^3 ax}{3a}$

589. $\int \sin ax\cos^n ax\mathrm{d}x = -\frac{\cos^{n+1}ax}{(n+1)a} \quad (n\neq -1)$

590. $\int \sin^2 ax\cos^2 ax\mathrm{d}x = -\frac{\sin 4ax}{32a}+\frac{x}{8}$

591. $\int \sin^m ax\cos^2 ax\mathrm{d}x = -\frac{\sin^{m+1}ax\cos ax}{(m+2)a}+\int\frac{\sin^m ax}{m+2}\mathrm{d}x \quad (m\neq -2)$

592. $\int \sin^2 ax\cos^n ax\mathrm{d}x = -\frac{\sin ax\cos^{n+1}ax}{(n+2)a}+\int\frac{\cos^n ax}{n+2}\mathrm{d}x \quad (n\neq -2)$

593. $\int \sin^3 ax\cos^3 ax\mathrm{d}x = -\frac{3\cos 2ax}{64a}+\frac{\cos ax}{192a}$

594. $\int \sin^m ax\cos^3 ax\mathrm{d}x = -\frac{\sin^{m+1}ax}{(m+1)a}-\frac{\sin^{m+3}ax}{(m+3)a} \quad (m\neq -1,-3)$

595. $\int \sin^3 ax\cos^n ax\mathrm{d}x = -\frac{\cos^{n+1}ax}{(n+1)a}+\frac{\cos^{n+3}ax}{(n+3)a} \quad (n\neq -1,-3)$

596. $\int \sin^m ax\cos^n ax\mathrm{d}x = \frac{\sin^{m+1}ax\cos^{n-1}ax}{(m+n)a}+\frac{n-1}{m+n}\int\sin^m ax\cos^{n-2}ax\mathrm{d}x$

$$= -\frac{\sin^{m-1}ax\cos^{n+1}ax}{(m+n)a}+\frac{m-1}{m+n}\int\sin^{m-2}ax\cos^n ax\mathrm{d}x$$

$(m>0,\ n>0)$

Ⅰ.1.3.3 含有$\frac{\sin^m ax}{\cos^n ax}$和$\frac{\cos^m ax}{\sin^n ax}$的积分

597. $\int \frac{\sin ax}{\cos ax}\mathrm{d}x = \int \tan ax\mathrm{d}x = -\frac{\ln|\cos ax|}{a}$

598. $\int \frac{\sin ax}{\cos^2 ax}\mathrm{d}x = \frac{1}{a\cos ax} = \frac{\sec ax}{a}$

599. $\int \frac{\sin ax}{\cos^n ax}\mathrm{d}x = \frac{1}{(n-1)a\cos^{n-1} ax} \quad (n \neq 1)$

600. $\int \frac{\sin^2 ax}{\cos ax}\mathrm{d}x = -\frac{\sin ax}{a} + \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right|$

601. $\int \frac{\sin^2 ax}{\cos^2 ax}\mathrm{d}x = \int \tan^2 ax\mathrm{d}x = \frac{\tan ax}{a} - x$

602. $\int \frac{\sin^2 ax}{\cos^n ax}\mathrm{d}x = \frac{\sin ax}{(n-1)a\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{\mathrm{d}x}{\cos^{n-2} ax} \quad (n \neq 1)$

603. $\int \frac{\sin^3 ax}{\cos ax}\mathrm{d}x = -\frac{\sin^2 ax}{2a} - \frac{1}{a}\ln|\cos ax|$

604. $\int \frac{\sin^3 ax}{\cos^2 ax}\mathrm{d}x = \frac{\cos ax}{a} + \frac{1}{a\ln|\cos ax|}$

605. $\int \frac{\sin^3 ax}{\cos^3 ax}\mathrm{d}x = \int \tan^3 ax\mathrm{d}x = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\ln|\cos ax|$

606. $\int \frac{\sin^3 ax}{\cos^n ax}\mathrm{d}x = \frac{1}{(n-1)a\cos^{n-1} ax} - \frac{1}{(n-3)a\cos^{n-3} ax} \quad (n \neq 1,3)$

607. $\int \frac{\sin^m ax}{\cos ax}\mathrm{d}x = -\frac{\sin^{m-1} ax}{(m-1)a} + \int \frac{\sin^{m-2} ax}{\cos ax}\mathrm{d}x \quad (m \neq 1)$

608. $\int \frac{\sin^{2m+1} ax}{\cos ax}\mathrm{d}x = -\frac{1}{a}\sum_{k=1}^{m} \frac{\sin^{2k} ax}{2k} - \frac{1}{a}\ln|\cos ax|$

609. $\int \frac{\sin^{2m} ax}{\cos ax}\mathrm{d}x = -\frac{1}{a}\sum_{k=1}^{m} \frac{\sin^{2k-1} ax}{2k-1} + \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right|$

610. $\int \frac{\sin^m ax}{\cos^n ax}\mathrm{d}x = \frac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \frac{m-n+2}{n-1}\int \frac{\sin^m ax}{\cos^{n-2} ax}\mathrm{d}x \quad (n \neq 1)$

$= -\frac{\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \frac{m-1}{m-n}\int \frac{\sin^{m-2} ax}{\cos^n ax}\mathrm{d}x \quad (m \neq n)$

611. $\int \frac{\sin^m ax}{\cos^{m+2} ax}\mathrm{d}x = \frac{\tan^{m+1} ax}{(m+1)a} \quad (m \neq -1)$

612. $\int \frac{\sin^p ax}{\cos^{2n+1} ax}\mathrm{d}x$

$= \frac{1}{a}\frac{\sin^{p+1} ax}{2n}\Big[\sec^{2n} ax$

$$+\sum_{k=1}^{n-1}\frac{(2n-p-1)(2n-p-3)\cdots(2n-p-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)}\sec^{2n-2k}ax\Big]$$

$$+\frac{(2n-p-1)(2n-p-3)\cdots(3-p)(1-p)}{2^n n!}\int\frac{\sin^p ax}{\cos ax}\mathrm{d}x$$

(这里，p 为任意实数，以下同)

613. $\int\frac{\sin^p ax}{\cos^{2n}ax}\mathrm{d}x$

$$=\frac{1}{a}\frac{\sin^{p+1}ax}{2n-1}\Big[\sec^{2n-1}ax$$

$$+\sum_{k=1}^{n-1}\frac{(2n-p-2)(2n-p-4)\cdots(2n-p-2k)}{(2n-3)(2n-5)\cdots(2n-2k-1)}\sec^{2n-2k-1}ax\Big]$$

$$+\frac{(2n-p-2)(2n-p-4)\cdots(2-p)(-p)}{(2n-1)!!}\int\sin^p ax\,\mathrm{d}x$$

614. $\int\frac{\cos ax}{\sin ax}\mathrm{d}x=\int\cot ax\,\mathrm{d}x=\frac{\ln|\sin ax|}{a}$

615. $\int\frac{\cos ax}{\sin^2 ax}\mathrm{d}x=-\frac{\csc ax}{a}=-\frac{1}{a\sin ax}$

616. $\int\frac{\cos ax}{\sin^n ax}\mathrm{d}x=-\frac{1}{(n-1)a\sin^{n-1}ax}\quad(n\neq 1)$

617. $\int\frac{\cos^2 ax}{\sin ax}\mathrm{d}x=\frac{\cos ax}{a}+\frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$

618. $\int\frac{\cos^2 ax}{\sin^2 ax}\mathrm{d}x=\int\cot^2 ax\,\mathrm{d}x=-\frac{1}{a}\cot ax-x$

619. $\int\frac{\cos^2 ax}{\sin^n ax}\mathrm{d}x=-\frac{\cos ax}{(n-1)a\sin^{n-1}ax}-\frac{1}{n-1}\int\frac{\mathrm{d}x}{\sin^{n-2}ax}\quad(n\neq 1)$

620. $\int\frac{\cos^3 ax}{\sin ax}\mathrm{d}x=\frac{\cos^2 ax}{2a}+\frac{1}{a}\ln|\sin ax|$

621. $\int\frac{\cos^3 ax}{\sin^2 ax}\mathrm{d}x=-\frac{\sin ax}{a}-\frac{1}{a\sin ax}$

622. $\int\frac{\cos^3 ax}{\sin^3 ax}\mathrm{d}x=\int\cot^3 ax\,\mathrm{d}x=-\frac{1}{2a}\cot^2 ax-\frac{1}{a}\ln|\sin ax|$

623. $\int\frac{\cos^3 ax}{\sin^n ax}\mathrm{d}x=-\frac{1}{(n-1)a\sin^{n-1}ax}+\frac{1}{(n-3)a\sin^{n-3}ax}\quad(n\neq 1,3)$

624. $\int\frac{\cos^m ax}{\sin ax}\mathrm{d}x=\frac{\cos^{m-1}ax}{(m-1)a}+\int\frac{\cos^{m-2}ax}{\sin ax}\mathrm{d}x\quad(m\neq 1)$

625. $\int\frac{\cos^{2m+1}ax}{\sin ax}\mathrm{d}x=\frac{1}{a}\sum_{k=1}^{m}\frac{\cos^{2k}ax}{2k}+\frac{1}{a}\ln|\sin ax|$

626. $\int\frac{\cos^{2m}ax}{\sin ax}\mathrm{d}x=\frac{1}{a}\sum_{k=1}^{m}\frac{\cos^{2k-1}ax}{2k-1}+\frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$

627. $\int\frac{\cos^m ax}{\sin^n ax}\mathrm{d}x=-\frac{\cos^{m+1}ax}{a(n-1)\sin^{n-1}ax}-\frac{m-n+2}{n-1}\int\frac{\cos^m ax}{\sin^{n-2}ax}\mathrm{d}x\quad(n\neq 1)$

$$= \frac{\cos^{m-1}ax}{a(m-n)\sin^{n-1}ax} + \frac{m-1}{m-n}\int \frac{\cos^{m-2}ax}{\sin^n ax}\mathrm{d}x \quad (m \neq n)$$

628. $\int \frac{\cos^n ax}{\sin^{n+2}ax}\mathrm{d}x = -\frac{\cot^{n+1}ax}{(n+1)a} \quad (n \neq -1)$

629. $\int \frac{\cos^p ax}{\sin^{2n+1}ax}\mathrm{d}x$

$$= -\frac{1}{a}\frac{\cos^{p+1}ax}{2n}\Big[\csc^{2n}ax + \sum_{k=1}^{n-1}\frac{(2n-p-1)(2n-p-3)\cdots(2n-p-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)}\csc^{2n-2k}ax\Big] + \frac{(2n-p-1)(2n-p-3)\cdots(3-p)(1-p)}{2^n n!}\int \frac{\cos^p ax}{\sin ax}\mathrm{d}x$$

630. $\int \frac{\cos^p ax}{\sin^{2n}ax}\mathrm{d}x$

$$= -\frac{1}{a}\frac{\cos^{p+1}ax}{2n-1}\Big[\csc^{2n-1}ax + \sum_{k=1}^{n-1}\frac{(2n-p-2)(2n-p-4)\cdots(2n-p-2k)}{(2n-3)(2n-5)\cdots(2n-2k-1)}\csc^{2n-2k-1}ax\Big] + \frac{(2n-p-2)(2n-p-4)\cdots(2-p)(-p)}{(2n-1)!!}\int \cos^p ax\mathrm{d}x$$

Ⅰ.1.3.4 含有 $x^m\sin^n ax$ 和 $x^m\cos^n ax$ 的积分

631. $\int x\sin ax\mathrm{d}x = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$

632. $\int x\sin^2 ax\mathrm{d}x = \frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax$

633. $\int x\sin^3 ax\mathrm{d}x = \frac{x}{12a}\cos 3ax - \frac{1}{36a^2}\sin 3ax - \frac{3x}{4a}\cos ax + \frac{3}{4a^2}\sin ax$

634. $\int x\sin^n ax\mathrm{d}x = x\int \sin^n ax\mathrm{d}x - \int\Big(\int \sin^n ax\mathrm{d}x\Big)\mathrm{d}x$

635. $\int x^2\sin ax\mathrm{d}x = \frac{2x}{a^2}\sin ax + \frac{2-a^2x^2}{a^3}\cos ax$

636. $\int x^2\sin^2 ax\mathrm{d}x = \frac{x^3}{6} - \Big(\frac{x^2}{4a} - \frac{1}{8a^3}\Big)\sin 2ax - \frac{x}{4a^2}\cos 2ax$

637. $\int x^2\sin^3 ax\mathrm{d}x = \frac{1}{4}\int x^2(3\sin ax - \sin 3ax)\mathrm{d}x$

638. $\int x^2\sin^n ax\mathrm{d}x$

$$= x^2\int \sin^n ax\mathrm{d}x - 2x\int\Big(\int \sin^n ax\mathrm{d}x\Big)\mathrm{d}x + 2\int\Big[\int\Big(\int \sin^n ax\mathrm{d}x\Big)\mathrm{d}x\Big]\mathrm{d}x$$

639. $\int x^3\sin ax\mathrm{d}x = \frac{3a^2x^2-6}{a^4}\sin ax + \frac{6x-a^2x^3}{a^3}\cos ax$

640. $\int x^3\sin^2 ax\mathrm{d}x = \frac{x^4}{8} - \frac{2a^2x^3-3x}{8a^3}\sin 2ax + \frac{6a^2x^2-3}{16a^4}\cos 2ax$

641. $\int x^3\sin^3 ax\mathrm{d}x = \frac{1}{4}\int x^3(3\sin ax - \sin 3ax)\mathrm{d}x$

642. $\int x^m\sin ax\mathrm{d}x = -\frac{1}{a}x^m\cos ax + \frac{m}{a}\int x^{m-1}\cos ax\mathrm{d}x$

$$= \cos ax\sum_{r=0}^{\left[\frac{m}{2}\right]}(-1)^{r+1}\frac{m!}{(m-2r)!}\frac{x^{m-2r}}{a^{2r+1}}$$

$$+ \sin ax\sum_{r=0}^{\left[\frac{m-1}{2}\right]}(-1)^r\frac{m!}{(m-2r-1)!}\frac{x^{m-2r-1}}{a^{2r+2}}$$

643. $\int x^m\sin^n ax\mathrm{d}x = x^m\int\sin^n ax\mathrm{d}x - mx^{m-1}\int\left(\int\sin^n ax\mathrm{d}x\right)\mathrm{d}x$

$$+ m(m-1)x^{m-2}\int\left[\int\left(\int\sin^n ax\mathrm{d}x\right)\mathrm{d}x\right]\mathrm{d}x - \cdots$$

（这里，级数末项为 x^{m-m} 乘以 $\sin^n ax$ 的一个多次积分）

644. $\int x^m\sin^2 x\mathrm{d}x = \frac{x^{m+1}}{2(m+1)} + \frac{m!}{4}\left[\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{(-1)^{k+1}x^{m-2k}}{2^{2k}(m-2k)!}\sin 2x\right.$

$$\left.+ \sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^{k+1}x^{m-2k-1}}{2^{2k+1}(m-2k-1)!}\cos 2x\right]$$

645. $\int x^m\sin^n x\mathrm{d}x = \frac{x^{m-1}\sin^{n-1}x}{n^2}(m\sin x - nx\cos x)$

$$+ \frac{n-1}{n}\int x^m\sin^{n-2}x\mathrm{d}x - \frac{m(m-1)}{n^2}\int x^{m-2}\sin^n x\mathrm{d}x$$

646. $\int x\cos ax\mathrm{d}x = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$

647. $\int x\cos^2 ax\mathrm{d}x = \frac{x^2}{4} + \frac{x}{4a}\sin 2ax + \frac{1}{8a^2}\cos 2ax$

648. $\int x\cos^3 ax\mathrm{d}x = \frac{x}{12a}\sin 3ax + \frac{1}{36a^2}\cos 3ax + \frac{3x}{4a}\sin ax + \frac{3}{4a^2}\cos ax$

649. $\int x\cos^n ax\mathrm{d}x = x\int\cos^n ax\mathrm{d}x - \int\left(\int\cos^n ax\mathrm{d}x\right)\mathrm{d}x$

650. $\int x^2\cos ax\mathrm{d}x = \frac{2x}{a^2}\cos ax + \frac{a^2x^2-2}{a^3}\sin ax$

651. $\int x^2\cos^2 ax\mathrm{d}x = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x}{4a^2}\cos 2ax$

652. $\int x^2\cos^3 ax\mathrm{d}x = \frac{1}{4}\int x^2(3\cos ax + \cos 3ax)\mathrm{d}x$

653. $\int x^2\cos^n ax\mathrm{d}x = x^2\int\cos^n ax\mathrm{d}x - 2x\int\left(\int\cos^n ax\mathrm{d}x\right)\mathrm{d}x + \int\left[\int\left(\int\cos^n ax\mathrm{d}x\right)\mathrm{d}x\right]\mathrm{d}x$

654. $\int x^3\cos ax\mathrm{d}x = \frac{3a^2x^2-6}{a^4}\cos ax + \frac{a^2x^3-6x}{a^3}\sin ax$

655. $\int x^3\cos^2 ax\mathrm{d}x = \frac{x^4}{8} + \frac{2a^2x^3-3x}{8a^3}\sin 2ax + \frac{6a^2x^2-3}{16a^4}\cos 2ax$

656. $\int x^3\cos^3 ax\mathrm{d}x = \frac{1}{4}\int x^3(3\cos ax + \cos 3ax)\mathrm{d}x$

657. $\int x^m\cos ax\mathrm{d}x = \frac{x^m}{a}\sin ax - \frac{m}{a}\int x^{m-1}\sin ax\mathrm{d}x$

$$= \sin ax\sum_{r=0}^{\left[\frac{m}{2}\right]}(-1)^r\frac{m!}{(m-2r)!}\frac{x^{m-2r}}{a^{2r+1}} + \cos ax\sum_{r=0}^{\left[\frac{m-1}{2}\right]}(-1)^r\frac{m!}{(m-2r-1)!}\frac{x^{m-2r-1}}{a^{2r+2}}$$

658. $\int x^m\cos^n ax\mathrm{d}x = x^m\int\cos^n ax\mathrm{d}x - mx^{m-1}\int\left(\int\cos^n ax\mathrm{d}x\right)\mathrm{d}x$

$$+ m(m-1)x^{m-2}\int\left[\int\left(\int\cos^n ax\mathrm{d}x\right)\mathrm{d}x\right]\mathrm{d}x - \cdots$$

（这里，级数末项为 x^{m-m} 乘以 $\cos^n ax$ 的一个多次积分）

659. $\int x^m\cos^2 x\mathrm{d}x = \frac{x^{m+1}}{2(m+1)} - \frac{m!}{4}\left[\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{(-1)^{k+1}x^{m-2k}}{2^{2k}(m-2k)!}\sin 2x + \sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^{k+1}x^{m-2k-1}}{2^{2k+1}(m-2k-1)!}\cos 2x\right]$

660. $\int x^m\cos^n x\mathrm{d}x = \frac{x^{m-1}\cos^{n-1}x}{n^2}(m\cos x - nx\sin x) + \frac{n-1}{n}\int x^m\cos^{n-2}x\mathrm{d}x - \frac{m(m-1)}{n^2}\int x^{m-2}\cos^n x\mathrm{d}x$

Ⅰ.1.3.5 含有 $\frac{\sin^n ax}{x^m}$，$\frac{x^m}{\sin^n ax}$，$\frac{\cos^n ax}{x^m}$，$\frac{x^m}{\cos^n ax}$ 的积分

661. $\int\frac{\sin ax}{x}\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n+1}}{(2n+1)(2n+1)!} = \mathrm{Si}(ax)$

［这里，$\mathrm{Si}(ax)$ 为正弦积分（见附录），以下同］

662. $\int\frac{\sin ax}{x^2}\mathrm{d}x = a\mathrm{Ci}(ax) - \frac{\sin ax}{x}$

［这里，$\mathrm{Ci}(ax)$ 为余弦积分（见附录），以下同］

663. $\int\frac{\sin ax}{x^m}\mathrm{d}x = \frac{\sin ax}{(1-m)x^{m-1}} + \frac{a}{m-1}\int\frac{\cos ax}{x^{m-1}}\mathrm{d}x \quad (m\neq 1)$

664. $\int \frac{\sin ax}{x^{2m}}\mathrm{d}x = \frac{(-1)^{m+1}}{x(2m-1)!}\left[\sum_{k=0}^{m-2}\frac{(-1)^k(2k+1)!a^{2m-2k-3}}{x^{2k+1}}\cos ax\right.$

$\left.+\sum_{k=0}^{m-1}\frac{(-1)^{k+1}(2k)!a^{2m-2k-2}}{x^{2k}}\sin ax\right]+\frac{(-1)^{m+1}a^{2m-1}}{(2m-1)!}\mathrm{ci}(ax)$

[这里，ci(x)为余弦积分(见附录)，以下同]

665. $\int \frac{\sin ax}{x^{2m+1}}\mathrm{d}x = \frac{(-1)^{m+1}}{x(2m)!}\left[\sum_{k=0}^{m-1}\frac{(-1)^k(2k)!a^{2m-2k-1}}{x^{2k}}\cos ax\right.$

$\left.+\sum_{k=0}^{m-1}\frac{(-1)^{k+1}(2k+1)!a^{2m-2k-2}}{x^{2k+1}}\sin ax\right]+\frac{(-1)^m a^{2m}}{(2m)!}\mathrm{si}(ax)$

[这里，si(x)为正弦积分(见附录)，以下同]

666. $\int \frac{\sin^2 ax}{x}\mathrm{d}x = \frac{1}{2}\ln|x| - \frac{1}{2}\mathrm{Ci}(2ax)$

667. $\int \frac{\sin^{2n} ax}{x}\mathrm{d}x = \binom{2n}{n}\frac{\ln x}{2^{2n}} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\mathrm{ci}[(2n-2k)ax]$

668. $\int \frac{\sin^{2n+1} ax}{x}\mathrm{d}x = \frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\mathrm{si}[(2n-2k+1)ax]$

669. $\int \frac{\sin^{2n-1} ax}{x^m}\mathrm{d}x$

$= \left(-\frac{1}{4}\right)^{n-1}\int\frac{1}{x^m}\left\{\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{k}\sin[(2n-2k-1)ax]\right\}\mathrm{d}x$

670. $\int \frac{\sin^{2n} ax}{x^m}\mathrm{d}x = 2\left(-\frac{1}{4}\right)^n\int\frac{1}{x^m}\left\{\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos[(2n-2k)ax]\right\}\mathrm{d}x$

$-\binom{2n}{n}\frac{1}{(m-1)4^n x^{m-1}}$

671. $\int \frac{x}{\sin ax}\mathrm{d}x = \frac{x}{a}\sum_{k=0}^{\infty}(-1)^{k-1}\frac{4^k-2}{(2k+1)!}B_{2k}(ax)^{2k} \quad (|ax|<\pi)$

[这里，B_{2k}为伯努利数(见附录)，以下同]

672. $\int \frac{x}{\sin^2 ax}\mathrm{d}x = \int x\csc^2 ax\mathrm{d}x = -\frac{x\cot ax}{a} + \frac{1}{a^2}\ln|\sin ax|$

673. $\int \frac{x}{\sin^n ax}\mathrm{d}x = \int x\csc^n ax\mathrm{d}x$

$= -\frac{x\cos ax}{a(n-1)\sin^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2}ax} + \frac{n-2}{n-1}\int\frac{x}{\sin^{n-2}ax}\mathrm{d}x$

674. $\int \frac{x^2}{\sin ax}\mathrm{d}x = \frac{x^2}{2a}\sum_{k=0}^{\infty}(-1)^{k-1}\frac{4^k-2}{(2k+1)!}B_{2k}(ax)^{2k} \quad (|ax|<\pi)$

675. $\int \frac{x^2}{\sin^2 ax}\mathrm{d}x = \frac{2x}{a^2}\left[1-\frac{ax}{2}\cot ax - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k+1)!}B_{2k}(2ax)^{2k}\right] \quad (|ax|<\pi)$

676. $\int \frac{x^m}{\sin ax}\mathrm{d}x = \frac{(ax)^m}{a^{m+1}}\sum_{k=0}^{\infty}(-1)^{k-1}\frac{4^k-2}{(2k)!}B_{2k}\frac{(ax)^{2k}}{m+2k} \quad (|ax|<\pi)$

677. $\int \frac{x^m}{\sin^n ax}\mathrm{d}x = -\frac{x^m}{(n-1)a}\frac{\cos ax}{\sin^{n-1}ax} - \frac{m}{(n-1)(n-2)a^2}\frac{x^{m-1}}{\sin^{n-2}ax}$

$+\frac{m(m-1)}{(n-1)(n-2)a^2}\int\frac{x^{m-2}}{\sin^{n-2}ax}\mathrm{d}x + \frac{n-2}{n-1}\int\frac{x^m}{\sin^{n-2}ax}\mathrm{d}x$

$(n\neq 1,2)$

678. $\int \frac{\cos ax}{x}\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n}}{2n(2n)!} = \mathrm{Ci}(ax)$

679. $\int \frac{\cos ax}{x^2}\mathrm{d}x = -\frac{\cos ax}{x} - a\mathrm{Si}(ax)$

680. $\int \frac{\cos ax}{x^m}\mathrm{d}x = -\frac{\cos ax}{(m-1)x^{m-1}} - \frac{a}{m-1}\int\frac{\sin ax}{x^{m-1}}\mathrm{d}x \quad (m\neq 1)$

681. $\int \frac{\cos ax}{x^{2m}}\mathrm{d}x = \frac{(-1)^{m+1}}{x(2m-1)!}\left[\sum_{k=0}^{m-1}\frac{(-1)^{k+1}(2k)!a^{2m-2k-2}}{x^{2k}}\cos ax\right.$

$\left.-\sum_{k=0}^{m-2}\frac{(-1)^{k+1}(2k+1)!a^{2m-2k-3}}{x^{2k+1}}\sin ax\right] + \frac{(-1)^m a^{2m-1}}{(2m-1)!}\mathrm{si}(ax)$

682. $\int \frac{\cos ax}{x^{2m+1}}\mathrm{d}x = \frac{(-1)^{m+1}}{x(2m)!}\left[\sum_{k=0}^{m-1}\frac{(-1)^{k+1}(2k+1)!a^{2m-2k-2}}{x^{2k+1}}\cos ax\right.$

$\left.-\sum_{k=0}^{m-1}\frac{(-1)^{k+1}(2k)!a^{2m-2k}}{x^{2k}}\sin ax\right] + \frac{(-1)^m a^{2m}}{(2m)!}\mathrm{ci}(ax)$

683. $\int \frac{\cos^2 ax}{x}\mathrm{d}x = \frac{1}{2}\ln|x| + \frac{1}{2}\mathrm{Ci}(2ax)$

684. $\int \frac{\cos^{2n} ax}{x}\mathrm{d}x = \binom{2n}{n}\frac{\ln x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\mathrm{ci}[(2n-2k)ax]$

685. $\int \frac{\cos^{2n+1} ax}{x}\mathrm{d}x = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}\mathrm{ci}[(2n-2k+1)ax]$

686. $\int \frac{\cos^{2n-1} ax}{x^m}\mathrm{d}x = \left(\frac{1}{4}\right)^{n-1}\int\frac{1}{x^m}\left\{\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos[(2n-2k-1)ax]\right\}\mathrm{d}x$

687. $\int \frac{\cos^{2n} ax}{x^m}\mathrm{d}x$

$= 2\left(\frac{1}{4}\right)^n\int\frac{1}{x^m}\left\{\sum_{k=0}^{n-1}\binom{2n}{k}\cos[(2n-2k)ax]\right\}\mathrm{d}x - \binom{2n}{n}\frac{1}{(m-1)4^n x^{m-1}}$

688. $\int \frac{x}{\cos ax}\mathrm{d}x = \frac{x^2}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|(ax)^{2k}}{(2k)!(k+1)} \quad \left(|ax|<\frac{\pi}{2}\right)$

[这里，E_{2k} 为欧拉数(见附录)，以下同]

689. $\int \frac{x}{\cos^2 ax}\mathrm{d}x = \int x\sec^2 ax\,\mathrm{d}x = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln|\cos ax|$

690. $\int \frac{x}{\cos^n ax}\mathrm{d}x = \int x\sec^n ax\mathrm{d}x$

$$= \frac{x\sin ax}{a(n-1)\cos^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2}ax} + \frac{n-2}{n-1}\int \frac{x}{\cos^{n-2}ax}\mathrm{d}x$$

691. $\int \frac{x^2}{\cos ax}\mathrm{d}x = x^3\sum_{k=0}^{\infty}\frac{|E_{2k}|(ax)^{2k}}{(2k)!(2k+3)} \quad \left(|ax| < \frac{\pi}{2}\right)$

692. $\int \frac{x^2}{\cos^2 ax}\mathrm{d}x = \frac{2x}{a^2}\left[\frac{ax}{2}\tan ax - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{4^k-1}{(2k+1)!}B_{2k}(2ax)^{2k}\right]$

$\left(|ax| < \frac{\pi}{2}\right)$

693. $\int \frac{x^m}{\cos ax}\mathrm{d}x = x^{m+1}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k)!}\frac{(ax)^{2k}}{(m+2k+1)} \quad \left(|ax| < \frac{\pi}{2}\right)$

694. $\int \frac{x^m}{\cos^n ax}\mathrm{d}x = \frac{x^m\sin ax}{(n-1)a\cos^{n-1}ax} - \frac{mx^{m-1}}{(n-1)(n-2)a^2\cos^{n-2}ax}$

$$+ \frac{m(m-1)}{(n-1)(n-2)a^2}\int \frac{x^{m-2}}{\cos^{n-2}ax}\mathrm{d}x + \frac{n-2}{n-1}\int \frac{x^m}{\cos^{n-2}ax}\mathrm{d}x$$

$(n \neq 1,2)$

Ⅰ.1.3.6 含有 $\sin ax\sin bx$，$\sin ax\cos bx$ 和 $\cos ax\cos bx$ 的积分

695. $\int \sin ax\sin bx\mathrm{d}x = -\frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$

696. $\int \sin ax\cos bx\mathrm{d}x = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$

697. $\int \cos ax\cos bx\mathrm{d}x = \frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$

698. $\int \sin ax\sin^2 bx\mathrm{d}x = \frac{\cos(a+2b)x}{4(a+2b)} - \frac{\cos(2b-a)x}{4(2b-a)} - \frac{\cos ax}{2a} \quad (a^2 \neq 4b^2)$

699. $\int \sin ax\cos^2 bx\mathrm{d}x = -\frac{\cos(a+2b)x}{4(a+2b)} + \frac{\cos(2b-a)x}{4(2b-a)} - \frac{\cos ax}{2a} \quad (a^2 \neq 4b^2)$

700. $\int \cos ax\sin^2 bx\mathrm{d}x = -\frac{\sin(a+2b)x}{4(a+2b)} - \frac{\sin(2b-a)x}{4(2b-a)} + \frac{\sin ax}{2a} \quad (a^2 \neq 4b^2)$

701. $\int \cos ax\cos^2 bx\mathrm{d}x = \frac{\sin(a+2b)x}{4(a+2b)} + \frac{\sin(2b-a)x}{4(2b-a)} + \frac{\sin ax}{2a} \quad (a^2 \neq 4b^2)$

702. $\int \sin ax\sin bx\sin cx\mathrm{d}x$

$$= \frac{\cos(a+b+c)x}{4(a+b+c)} - \frac{\cos(b+c-a)x}{4(b+c-a)} - \frac{\cos(c+a-b)x}{4(c+a-b)} - \frac{\cos(a+b-c)x}{4(a+b-c)}$$

$(a+b+c \neq 0,\ b+c-a \neq 0,\ c+a-b \neq 0,\ a+b-c \neq 0)$

703. $\int \sin ax\cos bx\cos cx\mathrm{d}x = -\frac{\cos(a+b+c)x}{4(a+b+c)} + \frac{\cos(b+c-a)x}{4(b+c-a)}$

$$-\frac{\cos(c+a-b)x}{4(c+a-b)}-\frac{\cos(a+b-c)x}{4(a+b-c)}$$

$(a+b+c\neq 0,\ b+c-a\neq 0,\ c+a-b\neq 0,\ a+b-c\neq 0)$

704. $\int \cos ax\sin bx\sin cx\,\mathrm{d}x=-\frac{\sin(a+b+c)x}{4(a+b+c)}-\frac{\sin(b+c-a)x}{4(b+c-a)}$

$$+\frac{\sin(c+a-b)x}{4(c+a-b)}+\frac{\sin(a+b-c)x}{4(a+b-c)}$$

$(a+b+c\neq 0,\ b+c-a\neq 0,\ c+a-b\neq 0,\ a+b-c\neq 0)$

705. $\int \cos ax\cos bx\cos cx\,\mathrm{d}x=\frac{\sin(a+b+c)x}{4(a+b+c)}+\frac{\sin(b+c-a)x}{4(b+c-a)}$

$$+\frac{\sin(c+a-b)x}{4(c+a-b)}+\frac{\sin(a+b-c)x}{4(a+b-c)}$$

$(a+b+c\neq 0,\ b+c-a\neq 0,\ c+a-b\neq 0,\ a+b-c\neq 0)$

706. $\int x\sin ax\sin bx\,\mathrm{d}x$

$$=-x\left[\frac{\sin(a+b)x}{2(a+b)}-\frac{\sin(a-b)x}{2(a-b)}\right]-\left[\frac{\cos(a+b)x}{2(a+b)^2}-\frac{\cos(a-b)x}{2(a-b)^2}\right]$$

$(a^2\neq b^2)$

707. $\int x\sin ax\cos bx\,\mathrm{d}x$

$$=-x\left[\frac{\cos(a+b)x}{2(a+b)}+\frac{\cos(a-b)x}{2(a-b)}\right]+\left[\frac{\sin(a+b)x}{2(a+b)^2}+\frac{\sin(a-b)x}{2(a-b)^2}\right]$$

$(a^2\neq b^2)$

708. $\int x\cos ax\cos bx\,\mathrm{d}x$

$$=x\left[\frac{\sin(a+b)x}{2(a+b)}+\frac{\sin(a-b)x}{2(a-b)}\right]+\left[\frac{\cos(a+b)x}{2(a+b)^2}+\frac{\cos(a-b)x}{2(a-b)^2}\right]$$

$(a^2\neq b^2)$

Ⅰ.1.3.7 含有$\frac{1}{\sin^m ax\cos^n ax}$的积分

709. $\int \frac{\mathrm{d}x}{\sin ax\cos ax}=\frac{1}{a}\ln|\tan ax|$

710. $\int \frac{\mathrm{d}x}{\sin^2 ax\cos ax}=-\frac{1}{a}\csc ax+\frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|$

711. $\int \frac{\mathrm{d}x}{\sin^m ax\cos ax}=-\frac{1}{a(m-1)\sin^{n-1}ax}+\int\frac{\mathrm{d}x}{\sin^{m-2}ax\cos ax}\quad(m\neq 1)$

712. $\int \frac{\mathrm{d}x}{\sin^{2m+1}ax\cos ax}=-\frac{1}{a}\sum_{k=1}^{m}\frac{1}{(2m-2k+2)\sin^{2m-2k+2}ax}+\frac{1}{a}\ln|\tan ax|$

713. $\int \frac{dx}{\sin^{2m}ax\cos ax}$

$$= -\frac{1}{a}\sum_{k=1}^{m}\frac{1}{(2m-2k+1)\sin^{2m-2k+1}ax} + \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right|$$

714. $\int \frac{dx}{\sin ax\cos^2 ax} = \frac{1}{a}\left(\sec ax + \ln\left|\tan\frac{ax}{2}\right|\right)$

715. $\int \frac{dx}{\sin ax\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1}ax} + \int\frac{dx}{\sin ax\cos^{n-2}ax}$

716. $\int \frac{dx}{\sin ax\cos^{2n+1}ax} = \frac{1}{a}\sum_{k=1}^{n}\frac{1}{(2n-2k+2)\cos^{2n-2k+2}ax} + \frac{1}{a}\ln|\tan ax|$

717. $\int \frac{dx}{\sin ax\cos^{2n}ax} = \frac{1}{a}\sum_{k=1}^{n}\frac{1}{(2n-2k+1)\cos^{2n-2k+1}ax} + \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$

718. $\int \frac{dx}{\sin^2 ax\cos^2 ax} = -\frac{2}{a}\cot 2ax$

719. $\int \frac{dx}{\sin^m ax\cos^n ax}$

$$= -\frac{1}{a(m-1)\sin^{m-1}ax\cos^{n-1}ax} + \frac{m+n-2}{m-1}\int\frac{dx}{\sin^{m-2}ax\cos^n ax}$$

$$= \frac{1}{a(n-1)\sin^{m-1}ax\cos^{n-1}ax} + \frac{m+n-2}{n-1}\int\frac{dx}{\sin^m ax\cos^{n-2}ax}$$

$(m \neq 1,\ n \neq 1)$

Ⅰ.1.3.8 含有 $1 \pm \sin ax$ 和 $1 \pm \cos ax$ 的积分

720. $\int \frac{dx}{1 \pm \sin ax} = \mp\frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right)$

721. $\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$

722. $\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$

723. $\int \frac{x}{1+\sin ax}dx = -\frac{x}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\cos\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right|$

724. $\int \frac{x}{1-\sin ax}dx = \frac{x}{a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right|$

725. $\int \frac{x}{1+\cos ax}dx = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right|$

726. $\int \frac{x}{1-\cos ax}dx = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right|$

727. $\int \frac{\sin ax}{1 \pm \sin ax}dx = \pm x + \frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right)$

728. $\int \frac{\sin ax}{1 \pm \cos ax} dx = \mp \frac{1}{a} \ln(1 \pm \cos ax)$

729. $\int \frac{\cos ax}{1 \pm \sin ax} dx = \pm \frac{1}{a} \ln(1 \pm \sin ax)$

730. $\int \frac{\cos ax}{1 + \cos ax} dx = -\frac{1}{a} \tan \frac{ax}{2} + x$

731. $\int \frac{\cos ax}{1 - \cos ax} dx = -\frac{1}{a} \cot \frac{ax}{2} - x$

732. $\int \frac{x + \sin ax}{1 + \cos ax} dx = \frac{x}{a} \tan \frac{ax}{2} + 2\left(\frac{1}{a^2} - \frac{1}{a}\right) \ln \left| \cos \frac{ax}{2} \right|$

733. $\int \frac{x - \sin ax}{1 - \cos ax} dx = -\frac{x}{a} \cot \frac{ax}{2} + 2\left(\frac{1}{a^2} - \frac{1}{a}\right) \ln \left| \sin \frac{ax}{2} \right|$

734. $\int \frac{dx}{\sin ax(1 \pm \sin ax)} = \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + \frac{1}{a} \ln \left| \tan \frac{ax}{2} \right|$

735. $\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \ln \left| \tan \frac{ax}{2} \right|$

736. $\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \ln \left| \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) \right|$

737. $\int \frac{dx}{\cos ax(1 + \cos ax)} = \frac{1}{a} \ln \left| \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) \right| - \frac{1}{a} \tan \frac{ax}{2}$

738. $\int \frac{dx}{\cos ax(1 - \cos ax)} = \frac{1}{a} \ln \left| \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) \right| - \frac{1}{a} \cot \frac{ax}{2}$

739. $\int \frac{\sin ax}{\cos ax(1 \pm \sin ax)} dx = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a} \ln \left| \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) \right|$

740. $\int \frac{\cos ax}{\sin ax(1 \pm \cos ax)} dx = \frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a} \ln \left| \tan \frac{ax}{2} \right|$

741. $\int \frac{\sin ax}{\cos ax(1 \pm \cos ax)} dx = \frac{1}{a} \ln \left| \sec ax \pm 1 \right|$

742. $\int \frac{\cos ax}{\sin ax(1 \pm \sin ax)} dx = -\frac{1}{a} \ln \left| \csc ax \pm 1 \right|$

743. $\int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a} \tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

744. $\int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a} \cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a} \cot^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

745. $\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a} \tan \frac{ax}{2} + \frac{1}{6a} \tan^3 \frac{ax}{2}$

746. $\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$

747. $\int \frac{\sin ax}{(1 + \sin ax)^2} dx = -\frac{1}{2a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a} \tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

748. $\int \frac{\sin ax}{(1 - \sin ax)^2} dx = -\frac{1}{2a} \cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a} \cot^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

749. $\int \frac{\cos ax}{(1+\cos ax)^2}\mathrm{d}x = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}$

750. $\int \frac{\cos ax}{(1-\cos ax)^2}\mathrm{d}x = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$

751. $\int \frac{\mathrm{d}x}{1+\cos ax \pm \sin ax} = \pm\frac{1}{a}\ln\left|1 \pm \tan\frac{ax}{2}\right|$

Ⅰ.1.3.9 含有 $a \pm b\sin cx$ 和 $a \pm b\cos cx$ 的积分

752. $$\int \frac{\mathrm{d}x}{a+b\sin cx} = \begin{cases} \frac{2}{c\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{cx}{2}+b}{\sqrt{a^2-b^2}} & (a^2>b^2) \\ \frac{1}{c\sqrt{b^2-a^2}}\ln\left|\frac{a\tan\frac{cx}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{cx}{2}+b+\sqrt{b^2-a^2}}\right| & (a^2<b^2) \end{cases}$$

753. $$\int \frac{\mathrm{d}x}{a+b\cos cx} = \begin{cases} \frac{2}{c\sqrt{a^2-b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{cx}{2}\right) & (a^2>b^2) \\ \frac{1}{c\sqrt{b^2-a^2}}\ln\left|\frac{\sqrt{b^2-a^2}\tan\frac{cx}{2}+a+b}{\sqrt{b^2-a^2}\tan\frac{cx}{2}-a-b}\right| & (a^2<b^2) \end{cases}$$

754. $\int \frac{\mathrm{d}x}{(a+b\sin cx)^2} = \frac{b\cos cx}{c(a^2-b^2)(a+b\sin cx)} + \frac{a}{a^2-b^2}\int\frac{\mathrm{d}x}{a+b\sin cx}$

755. $\int \frac{\mathrm{d}x}{(a+b\cos cx)^2} = \frac{b\sin cx}{c(b^2-a^2)(a+b\cos cx)} - \frac{a}{b^2-a^2}\int\frac{\mathrm{d}x}{a+b\cos cx}$

756. $$\int \frac{\mathrm{d}x}{(a+b\sin cx)^n} = \frac{b\cos cx}{c(n-1)(a^2-b^2)(a+b\sin cx)^{n-1}} + \frac{(2n-3)a}{(n-1)(a^2-b^2)}\int\frac{\mathrm{d}x}{(a+b\sin cx)^{n-1}} - \frac{n-2}{(n-1)(a^2-b^2)}\int\frac{\mathrm{d}x}{(a+b\sin cx)^{n-2}} \quad (a^2\neq b^2,\ n\neq 1)$$

757. $$\int \frac{\mathrm{d}x}{(a+b\cos cx)^n} = -\frac{b\sin cx}{c(n-1)(a^2-b^2)(a+b\cos cx)^{n-1}} + \frac{(2n-3)a}{(n-1)(a^2-b^2)}\int\frac{\mathrm{d}x}{(a+b\cos cx)^{n-1}} - \frac{n-2}{(n-1)(a^2-b^2)}\int\frac{\mathrm{d}x}{(a+b\cos cx)^{n-2}} \quad (a^2\neq b^2,\ n\neq 1)$$

758. $\int \frac{\sin cx}{a+b\sin cx}\mathrm{d}x = \frac{x}{b} - \frac{a}{b}\int\frac{\mathrm{d}x}{a+b\sin cx}$

759. $\int \frac{\cos cx}{a+b\cos cx}\mathrm{d}x = \frac{x}{b} - \frac{a}{b}\int\frac{\mathrm{d}x}{a+b\cos cx}$

760. $\int \frac{\sin cx}{a \pm b\cos cx}dx = \mp \frac{1}{bc}\ln|a \pm b\cos cx|$

761. $\int \frac{\cos cx}{a \pm b\sin cx}dx = \pm \frac{1}{bc}\ln|a \pm b\sin cx|$

762. $\int \frac{\sin cx}{(a + b\sin cx)^2}dx = -\frac{a\cos cx}{c(a^2 - b^2)(a + b\sin cx)} + \frac{b}{a^2 - b^2}\int \frac{dx}{a + b\sin cx}$
$(a^2 \neq b^2)$

763. $\int \frac{\cos cx}{(a + b\cos cx)^2}dx = \frac{a\sin cx}{c(a^2 - b^2)(a + b\cos cx)} - \frac{b}{a^2 - b^2}\int \frac{dx}{a + b\cos cx}$
$(a^2 \neq b^2)$

764. $\int \frac{\sin cx}{(a \pm b\cos cx)^2}dx = \pm \frac{1}{bc(a \pm b\cos cx)}$

765. $\int \frac{\cos cx}{(a \pm b\sin cx)^2}dx = \mp \frac{1}{bc(a \pm b\sin cx)}$

766. $\int \frac{\sin cx}{(a \pm b\cos cx)^n}dx = \pm \frac{1}{bc(n-1)(a \pm b\cos cx)^{n-1}} \quad (n \neq 0,1)$

767. $\int \frac{\cos cx}{(a \pm b\sin cx)^n}dx = \mp \frac{1}{bc(n-1)(a \pm b\sin cx)^{n-1}} \quad (n \neq 0,1)$

768. $\int \frac{dx}{\sin cx(a + b\sin cx)} = \frac{1}{ac}\ln\left|\tan\frac{cx}{2}\right| - \frac{b}{a}\int \frac{dx}{a + b\sin cx}$

769. $\int \frac{dx}{\cos cx(a + b\cos cx)} = \frac{1}{ac}\ln\left|\tan\left(\frac{\pi}{4} + \frac{cx}{2}\right)\right| - \frac{b}{a}\int \frac{dx}{a + b\cos cx}$

770. $\int \frac{dx}{a^2 + b^2 - 2ab\cos cx} = \frac{2}{c(a^2 - b^2)}\arctan\left(\frac{a + b}{a - b}\tan\frac{cx}{2}\right)$

Ⅰ.1.3.10 含有 $1 \pm b\sin^2 ax$，$1 \pm b\cos^2 ax$ 和 $c^2 \pm b^2\sin^2 ax$，$c^2 \pm b^2\cos^2 ax$ 的积分

771. $\int \frac{dx}{1 + b\sin^2 ax} = \frac{1}{a\sqrt{1 + b}}\arctan(\sqrt{1 + b}\tan ax)$

772. $\int \frac{dx}{1 - b\sin^2 ax} = \begin{cases} \frac{1}{a\sqrt{1 - b}}\arctan(\sqrt{1 - b}\tan ax) & (0 < b < 1) \\ \frac{1}{2a\sqrt{b - 1}}\ln\left|\frac{\sqrt{b - 1}\tan ax + 1}{\sqrt{b - 1}\tan ax - 1}\right| & (b > 1) \end{cases}$

773. $\int \frac{dx}{1 + b\cos^2 ax} = \frac{1}{a\sqrt{1 + b}}\arctan\frac{\tan ax}{\sqrt{1 + b}} \quad (b > 0)$

774. $\int \frac{dx}{1 - b\cos^2 ax} = \begin{cases} \frac{1}{a\sqrt{1 - b}}\arctan\frac{\tan ax}{\sqrt{1 - b}} & (0 < b < 1) \\ \frac{1}{2a\sqrt{b - 1}}\ln\left|\frac{\tan ax - \sqrt{b - 1}}{\tan ax + \sqrt{b - 1}}\right| & (b > 1) \end{cases}$

775. $\int \frac{dx}{(1+b\sin^2 ax)^2}$

$$= \frac{b\sin 2ax}{4a(1+b)(1+b\sin^2 ax)} + \frac{2+b}{2a\sqrt{(1+b)^3}}\arctan(\sqrt{1+b}\tan ax)$$

$(b>0)$

776. $\int \frac{dx}{(1-b\sin^2 ax)^2} = -\frac{b\sin 2ax}{4a(1-b)(1-b\sin^2 ax)} + \frac{2-b}{2a(1-b)}$

$$\cdot\begin{cases} \frac{1}{\sqrt{1-b}}\arctan(\sqrt{1-b}\tan ax) & (0<b<1) \\ \frac{1}{2\sqrt{b-1}}\ln\left|\frac{\sqrt{b-1}\tan ax+1}{\sqrt{b-1}\tan ax-1}\right| & (b>1) \end{cases}$$

777. $\int \frac{dx}{(1+b\cos^2 ax)^2}$

$$= -\frac{b\sin 2ax}{4a(1+b)(1+b\cos^2 ax)} + \frac{2+b}{2a\sqrt{(1+b)^3}}\arctan\frac{\tan ax}{\sqrt{1+b}} \quad (b>0)$$

778. $\int \frac{dx}{(1-b\cos^2 ax)^2} = \frac{b\sin 2ax}{4a(1-b)(1-b\cos^2 ax)} + \frac{2-b}{2a(1-b)}$

$$\cdot\begin{cases} \frac{1}{\sqrt{1-b}}\arctan\frac{\tan ax}{\sqrt{1-b}} & (0<b<1) \\ \frac{1}{2\sqrt{b-1}}\ln\left|\frac{\tan ax-\sqrt{b-1}}{\tan ax+\sqrt{b-1}}\right| & (b>1) \end{cases}$$

779. $\int \frac{\sin^2 ax}{1+b\sin^2 ax}dx = \frac{x}{b} - \frac{1}{ab\sqrt{1+b}}\arctan(\sqrt{1+b}\tan ax) \quad (b>0)$

780. $\int \frac{\sin^2 ax}{1-b\sin^2 ax}dx = \begin{cases} \frac{1}{ab\sqrt{1-b}}\arctan(\sqrt{1-b}\tan ax) - \frac{x}{b} & (0<b<1) \\ \frac{1}{2ab\sqrt{b-1}}\ln\left|\frac{\sqrt{b-1}\tan ax+1}{\sqrt{b-1}\tan ax-1}\right| - \frac{x}{b} & (b>1) \end{cases}$

781. $\int \frac{\cos^2 ax}{1+b\sin^2 ax}dx = -\frac{x}{b} + \frac{\sqrt{1+b}}{ab}\arctan(\sqrt{1+b}\tan ax) \quad (b>0)$

782. $\int \frac{\cos^2 ax}{1-b\sin^2 ax}dx = \begin{cases} -\frac{\sqrt{1-b}}{ab}\arctan(\sqrt{1-b}\tan ax) + \frac{x}{b} & (0<b<1) \\ \frac{\sqrt{b-1}}{2ab}\ln\left|\frac{\sqrt{b-1}\tan ax+1}{\sqrt{b-1}\tan ax-1}\right| + \frac{x}{b} & (b>1) \end{cases}$

783. $\int \frac{\sin^2 ax}{1+b\cos^2 ax}dx = -\frac{x}{b} + \frac{\sqrt{1+b}}{ab}\arctan\frac{\tan ax}{\sqrt{1+b}} \quad (b>0)$

784. $\int \frac{\sin^2 ax}{1-b\cos^2 ax}\mathrm{d}x = \begin{cases} -\frac{\sqrt{1-b}}{ab}\arctan\frac{\tan ax}{\sqrt{1-b}} + \frac{x}{b} & (0<b<1) \\ \frac{\sqrt{b-1}}{2ab}\ln\left|\frac{\tan ax-\sqrt{b-1}}{\tan ax+\sqrt{b-1}}\right| + \frac{x}{b} & (b>1) \end{cases}$

785. $\int \frac{\cos^2 ax}{1+b\cos^2 ax}\mathrm{d}x = \frac{x}{b} - \frac{1}{ab\sqrt{1+b}}\arctan\frac{\tan ax}{\sqrt{1+b}} \quad (b>0)$

786. $\int \frac{\cos^2 ax}{1-b\cos^2 ax}\mathrm{d}x = \begin{cases} \frac{1}{ab\sqrt{1-b}}\arctan\frac{\tan ax}{\sqrt{1-b}} - \frac{x}{b} & (0<b<1) \\ \frac{1}{2ab\sqrt{b-1}}\ln\left|\frac{\tan ax-\sqrt{b-1}}{\tan ax+\sqrt{b-1}}\right| - \frac{x}{b} & (b>1) \end{cases}$

787. $\int \frac{\sin ax\cos ax}{1\pm b\sin^2 ax}\mathrm{d}x = \pm\frac{1}{ab}\ln\sqrt{1\pm b\sin^2 ax} \quad (b>0)$

788. $\int \frac{\sin ax\cos ax}{1\pm b\cos^2 ax}\mathrm{d}x = \mp\frac{1}{ab}\ln\sqrt{1\pm b\cos^2 ax} \quad (b>0)$

789. $\int \frac{\mathrm{d}x}{c^2+b^2\sin^2 ax} = \frac{1}{ac\sqrt{c^2+b^2}}\arctan\frac{\sqrt{c^2+b^2}\tan ax}{c}$

790. $\int \frac{\mathrm{d}x}{c^2-b^2\sin^2 ax} = \begin{cases} \frac{1}{ac\sqrt{c^2-b^2}}\arctan\frac{\sqrt{c^2-b^2}\tan ax}{c} & (c^2>b^2) \\ \frac{1}{2ac\sqrt{b^2-c^2}}\ln\left|\frac{\sqrt{b^2-c^2}\tan ax+c}{\sqrt{b^2-c^2}\tan ax-c}\right| & (c^2<b^2) \end{cases}$

791. $\int \frac{\mathrm{d}x}{c^2+b^2\cos^2 ax} = \frac{1}{ac\sqrt{c^2+b^2}}\arctan\frac{c\tan ax}{\sqrt{c^2+b^2}}$

792. $\int \frac{\mathrm{d}x}{c^2-b^2\cos^2 ax} = \begin{cases} \frac{1}{ac\sqrt{c^2-b^2}}\arctan\frac{c\tan ax}{\sqrt{c^2-b^2}} & (c^2>b^2) \\ \frac{1}{2ac\sqrt{b^2-c^2}}\ln\left|\frac{c\tan ax-\sqrt{b^2-c^2}}{c\tan ax+\sqrt{b^2-c^2}}\right| & (c^2<b^2) \end{cases}$

793. $\int \frac{\sin^2 ax}{c^2+b^2\cos^2 ax}\mathrm{d}x = \frac{1}{ab^2}\sqrt{\frac{c^2+b^2}{c^2}}\arctan\left(\sqrt{\frac{c^2}{c^2+b^2}}\tan ax\right) - \frac{x}{b^2}$

794. $\int \frac{\cos^2 ax}{c^2+b^2\sin^2 ax}\mathrm{d}x = \frac{1}{ab^2}\sqrt{\frac{c^2+b^2}{c^2}}\arctan\left(\sqrt{\frac{c^2+b^2}{c^2}}\tan ax\right) - \frac{x}{b^2}$

Ⅰ.1.3.11 含有 $p\sin ax + q\cos ax$ 的积分

795. $\int \frac{\mathrm{d}x}{p\sin ax+q\cos ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\left|\tan\left(\frac{ax}{2}+\frac{1}{2}\arctan\frac{q}{p}\right)\right|$

796. $\int \frac{\mathrm{d}x}{(p\sin ax+q\cos ax)^2} = -\frac{1}{a(p^2+q^2)}\tan\left(ax+\arctan\frac{q}{p}\right)$

797. $\int \frac{dx}{(p\sin ax + q\cos ax)^n} = -\frac{\cos\left(ax + \arctan\frac{q}{p}\right)}{a(n-1)\sqrt{(p^2+q^2)^n}\sin^{n-1}\left(ax + \arctan\frac{q}{p}\right)}$

$$+\frac{n-2}{(n-1)\sqrt{(p^2+q^2)^n}}\int\frac{d\left(ax+\arctan\frac{q}{p}\right)}{\sin^{n-2}\left(ax+\arctan\frac{q}{p}\right)}$$

798. $\int \frac{\sin ax}{p\sin ax + q\cos ax}dx = \frac{1}{a(p^2+q^2)}(pax - q\ln|p\sin ax + q\cos ax|)$

799. $\int \frac{\sin ax}{\sin ax \pm \cos ax}dx = \frac{1}{2a}(ax \mp \ln|\sin ax \pm \cos ax|)$

800. $\int \frac{\cos ax}{p\sin ax + q\cos ax}dx = \frac{1}{a(p^2+q^2)}(pax + q\ln|p\sin ax + q\cos ax|)$

801. $\int \frac{\cos ax}{\sin ax \pm \cos ax}dx = \frac{1}{2a}(\ln|\sin ax \pm \cos ax| \pm ax)$

802. $\int \frac{p + q\sin ax}{\sin ax(1 \pm \cos ax)}dx = \frac{p}{2a}\left(\ln\left|\tan\frac{ax}{2}\right| \pm \frac{1}{1 \pm \cos ax}\right) + q\int\frac{dx}{1 \pm \cos ax}$

803. $\int \frac{p + q\sin ax}{\cos ax(1 \pm \cos ax)}dx$

$$= \frac{p}{a}\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| + \frac{q}{a}\ln\left|\frac{1 \pm \cos ax}{\cos ax}\right| - p\int\frac{dx}{1 \pm \cos ax}$$

804. $\int \frac{p + q\cos ax}{\sin ax(1 \pm \sin ax)}dx = \frac{p}{a}\ln\left|\tan\frac{ax}{2}\right| - \frac{q}{a}\ln\left|\frac{1 \pm \sin ax}{\sin ax}\right| - p\int\frac{dx}{1 \pm \sin ax}$

805. $\int \frac{p + q\cos ax}{\cos ax(1 \pm \sin ax)}dx = \frac{p}{2a}\left[\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| \mp \frac{1}{1 \pm \sin ax}\right] + q\int\frac{dx}{1 \pm \sin ax}$

Ⅰ.1.3.12 含有 $p^2\sin^2 ax \pm q^2\cos^2 ax$ 的积分

806. $\int(p^2\sin^2 ax \pm q^2\cos^2 ax)dx = \frac{(p^2 \pm q^2)x}{2} - \frac{(p^2 \mp q^2)\sin 2ax}{4a}$

807. $\int \frac{dx}{p^2\sin^2 ax + q^2\cos^2 ax} = \frac{1}{apq}\arctan\left(\frac{p}{q}\tan ax\right)$

808. $\int \frac{dx}{p^2\sin^2 ax - q^2\cos^2 ax} = \frac{1}{2apq}\ln\left|\frac{p\tan ax - q}{p\tan ax + q}\right|$

809. $\int \frac{dx}{(p^2\sin^2 ax \pm q^2\cos^2 ax)^2} = \frac{1}{2ap^3q^3}[(p^2 \pm q^2)u \pm (p^2 \mp q^2)\sin u\cos u]$

$\left[这里, u = \arctan\left(\frac{p}{q}\tan ax\right)\right]$

810. $\int \frac{dx}{(p^2\sin^2 ax \pm q^2\cos^2 ax)^n} = \frac{1}{a(pq)^{2n-1}}\int(p^2\sin^2 u \pm q^2\cos^2 u)^{n-1}du$

$\left[\text{这里}, u = \arctan\left(\frac{p}{q}\tan ax\right)\right]$

811. $\int \sin ax\cos ax(p^2\sin^2 ax \pm q^2\cos^2 ax)\mathrm{d}x = \frac{1}{4a}(p^2\sin^4 ax \mp q^2\cos^4 ax)$

812. $\int \frac{\sin ax\cos ax}{p^2\sin^2 ax \pm q^2\cos^2 ax}\mathrm{d}x = \frac{1}{2a(p^2 \mp q^2)}\ln|p^2\sin^2 ax \pm q^2\cos^2 ax|$

$(p^2 - q^2 \neq 0)$

813. $\int \frac{\sin ax\cos ax}{\sqrt{p^2\sin^2 ax \pm q^2\cos^2 ax}}\mathrm{d}x = \frac{1}{a(p^2 \mp q^2)}\sqrt{p^2\sin^2 ax \pm q^2\cos^2 ax}$

Ⅰ.1.3.13 含有 $\sin^m x, \cos^m x$ 与 $\sin nx, \cos nx$ 组合的积分

814. $\int \sin(n+1)x\sin^{n-1}x\,\mathrm{d}x = \frac{1}{n}\sin^n x\sin nx$

815. $\int \sin(n+1)x\cos^{n-1}x\,\mathrm{d}x = -\frac{1}{n}\cos^n x\cos nx$

816. $\int \cos(n+1)x\sin^{n-1}x\,\mathrm{d}x = \frac{1}{n}\sin^n x\cos nx$

817. $\int \cos(n+1)x\cos^{n-1}x\,\mathrm{d}x = \frac{1}{n}\cos^n x\sin nx$

818. $\int \frac{\sin(2n+1)x}{\sin x}\mathrm{d}x = 2\sum_{k=1}^{n}\frac{\sin 2kx}{2k} + x$

819. $\int \frac{\sin 2nx}{\sin x}\mathrm{d}x = 2\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1}$

820. $\int \frac{\cos(2n+1)x}{\sin x}\mathrm{d}x = 2\sum_{k=1}^{n}\frac{\cos 2kx}{2k} + \ln|\sin x|$

821. $\int \frac{\cos 2nx}{\sin x}\mathrm{d}x = 2\sum_{k=1}^{n}\frac{\cos(2k-1)x}{2k-1} + \ln\left|\tan\frac{x}{2}\right|$

822. $\int \frac{\sin(2n+1)x}{\cos x}\mathrm{d}x = 2\sum_{k=1}^{n}(-1)^{n-k+1}\frac{\cos 2kx}{2k} + (-1)^{n+1}\ln|\cos x|$

823. $\int \frac{\sin 2nx}{\cos x}\mathrm{d}x = 2\sum_{k=1}^{n}(-1)^{n-k+1}\frac{\cos(2k-1)x}{2k-1}$

824. $\int \frac{\cos(2n+1)x}{\cos x}\mathrm{d}x = 2\sum_{k=1}^{n}(-1)^{n-k}\frac{\sin 2kx}{2k} + (-1)^n x$

825. $\int \frac{\cos 2nx}{\cos x}\mathrm{d}x = 2\sum_{k=1}^{n}(-1)^{n-k}\frac{\sin(2k-1)x}{2k-1} + (-1)^n\ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$

Ⅰ.1.3.14 含有 $\sin(ax+b)$ 和 $\cos(cx+d)$ 的积分

826. $\int \sin(ax+b)\mathrm{d}x = -\frac{1}{a}\cos(ax+b)$

827. $\int \cos(ax+b)\mathrm{d}x = \frac{1}{a}\sin(ax+b)$

828. $\int \sin(ax+b)\sin(cx+d)\mathrm{d}x$

$$= -\frac{\sin[(a+c)x+(b+d)]}{2(a+c)} + \frac{\sin[(a-c)x+(b-d)]}{2(a-c)} \quad (a^2 \neq c^2)$$

829. $\int \sin(ax+b)\cos(cx+d)\mathrm{d}x$

$$= -\frac{\cos[(a+c)x+(b+d)]}{2(a+c)} - \frac{\cos[(a-c)x+(b-d)]}{2(a-c)} \quad (a^2 \neq c^2)$$

830. $\int \cos(ax+b)\sin(cx+d)\mathrm{d}x$

$$= -\frac{\cos[(a+c)x+(b+d)]}{2(a+c)} + \frac{\cos[(a-c)x+(b-d)]}{2(a-c)} \quad (a^2 \neq c^2)$$

831. $\int \cos(ax+b)\cos(cx+d)\mathrm{d}x$

$$= \frac{\sin[(a+c)x+(b+d)]}{2(a+c)} + \frac{\sin[(a-c)x+(b-d)]}{2(a-c)} \quad (a^2 \neq c^2)$$

832. $\int \sin^2(ax+b)\mathrm{d}x = \frac{1}{2}x - \frac{1}{4a}\sin 2(ax+b)$

833. $\int \cos^2(ax+b)\mathrm{d}x = \frac{1}{2}x + \frac{1}{4a}\sin 2(ax+b)$

834. $\int \mathrm{e}^{px}\sin(ax+b)\mathrm{d}x = \frac{\mathrm{e}^{px}}{p^2+a^2}[p\sin(ax+b) - a\cos(ax+b)]$

835. $\int \mathrm{e}^{px}\cos(ax+b)\mathrm{d}x = \frac{\mathrm{e}^{px}}{p^2+a^2}[a\sin(ax+b) + p\cos(ax+b)]$

836. $\int [\mathrm{e}^{px}\sin(ax+b)]^2\mathrm{d}x = \frac{\mathrm{e}^{2px}}{4}\left[\frac{1}{p} - \frac{a\sin 2(ax+b) + p\cos 2(ax+b)}{p^2+a^2}\right]$

837. $\int [\mathrm{e}^{px}\cos(ax+b)]^2\mathrm{d}x = \frac{\mathrm{e}^{2px}}{4}\left[\frac{1}{p} + \frac{a\sin 2(ax+b) + p\cos 2(ax+b)}{p^2+a^2}\right]$

Ⅰ.1.3.15 含有 $\sqrt{1\pm\sin ax}$ 和 $\sqrt{1\pm\cos ax}$ 的积分

838. $\int \sqrt{1+\sin ax}\,\mathrm{d}x = -\frac{2\sqrt{2}}{a}\cos\left(\frac{ax}{2}+\frac{\pi}{4}\right)$

839. $\int \sqrt{1+\cos ax}\,\mathrm{d}x = \dfrac{2\sin ax}{a\sqrt{1+\cos ax}} = \dfrac{2\sqrt{2}}{a}\sin\dfrac{ax}{2}$

840. $\int \sqrt{1-\sin ax}\,\mathrm{d}x = \dfrac{2\sqrt{2}}{a}\sin\left(\dfrac{ax}{2}+\dfrac{\pi}{4}\right)$

841. $\int \sqrt{1-\cos ax}\,\mathrm{d}x = -\dfrac{2\sin ax}{a\sqrt{1-\cos ax}} = -\dfrac{2\sqrt{2}}{a}\cos\dfrac{ax}{2}$

842. $\int \dfrac{\mathrm{d}x}{\sqrt{1+\sin ax}} = \dfrac{\sqrt{2}}{a}\ln\left|\tan\left(\dfrac{ax}{4}+\dfrac{\pi}{8}\right)\right|$

843. $\int \dfrac{\mathrm{d}x}{\sqrt{1+\cos ax}} = \dfrac{\sqrt{2}}{a}\ln\left|\tan\left(\dfrac{ax}{4}+\dfrac{\pi}{4}\right)\right|$

844. $\int \dfrac{\mathrm{d}x}{\sqrt{1-\sin ax}} = \dfrac{\sqrt{2}}{a}\ln\left|\tan\left(\dfrac{ax}{4}-\dfrac{\pi}{8}\right)\right|$

845. $\int \dfrac{\mathrm{d}x}{\sqrt{1-\cos ax}} = \dfrac{\sqrt{2}}{a}\ln\left|\tan\dfrac{ax}{4}\right|$

846. $\int \dfrac{\sin ax}{\sqrt{1\pm\cos ax}}\mathrm{d}x = \mp\dfrac{1}{2a}\sqrt{1\pm\cos ax}$

847. $\int \dfrac{\cos ax}{\sqrt{1\pm\sin ax}}\mathrm{d}x = \pm\dfrac{1}{2a}\sqrt{1\pm\sin ax}$

Ⅰ.1.3.16 含有 $\sqrt{1\pm b^2\sin^2 ax}$ 和 $\sqrt{1\pm b^2\cos^2 ax}$ 的积分

848. $\int \sqrt{1+b^2\sin^2 ax}\,\mathrm{d}x = -\dfrac{\sqrt{1+b^2}}{a}\cdot \mathrm{E}\left(\dfrac{p}{q},\dfrac{\pi}{2}-ax\right)$

[这里，$\mathrm{E}(k,\varphi)$ 为第二类椭圆积分(见附录)，以下同]

849. $\int \sqrt{1-b^2\sin^2 ax}\,\mathrm{d}x = \dfrac{1}{a}\cdot \mathrm{E}(b,ax) \quad (b^2<1)$

850. $\int \sqrt{1+b^2\cos^2 ax}\,\mathrm{d}x = \dfrac{\sqrt{1+b^2}}{a}\cdot \mathrm{E}\left(\sqrt{\dfrac{b^2}{1+b^2}},ax\right)$

851. $\int \sqrt{1-b^2\cos^2 ax}\,\mathrm{d}x = -\dfrac{\sqrt{1-b^2}}{a}\cdot \mathrm{E}\left(\sqrt{\dfrac{b^2}{1-b^2}},\dfrac{\pi}{2}-ax\right) \quad (b^2<1)$

852. $\int \dfrac{\mathrm{d}x}{\sqrt{1+b^2\sin^2 ax}} = -\dfrac{b}{a}\cdot \mathrm{F}\left(\dfrac{p}{q},\dfrac{\pi}{2}-ax\right)$

[这里，$\mathrm{F}(k,\varphi)$ 为第一类椭圆积分(见附录)，以下同]

853. $\int \dfrac{\mathrm{d}x}{\sqrt{1-b^2\sin^2 ax}} = \dfrac{1}{a}\cdot \mathrm{F}(b,ax) \quad (b^2<1)$

854. $\int \dfrac{\mathrm{d}x}{\sqrt{1+b^2\cos^2 ax}} = \dfrac{1}{a\sqrt{1+b^2}}\cdot \mathrm{F}\left(\sqrt{\dfrac{b^2}{1+b^2}},ax\right)$

855. $\int \frac{\mathrm{d}x}{\sqrt{1-b^2\cos^2 ax}} = -\frac{1}{a\sqrt{1-b^2}} \cdot \mathrm{F}\left(\sqrt{\frac{b^2}{1-b^2}}, \frac{\pi}{2} - ax\right) \quad (b^2 < 1)$

856. $\int \sin ax \sqrt{1+b^2\sin^2 ax}\,\mathrm{d}x = -\frac{\cos ax}{2a}\sqrt{1+b^2\sin^2 ax} - \frac{1+b^2}{2ab}\arcsin\frac{b\cos ax}{\sqrt{1+b^2}}$

857. $\int \cos ax \sqrt{1+b^2\sin^2 ax}\,\mathrm{d}x$

$$= \frac{\sin ax}{2a}\sqrt{1+b^2\sin^2 ax} + \frac{1}{2ab}\ln(b\sin ax + \sqrt{1+b^2\sin^2 ax})$$

858. $\int \sin ax \sqrt{1-b^2\sin^2 ax}\,\mathrm{d}x$

$$= -\frac{\cos ax}{2a}\sqrt{1-b^2\sin^2 ax} - \frac{1-b^2}{2ab}\ln(b\cos ax + \sqrt{1-b^2\sin^2 ax}) \quad (b^2 < 1)$$

859. $\int \cos ax \sqrt{1-b^2\sin^2 ax}\,\mathrm{d}x = \frac{\sin ax}{2a}\sqrt{1-b^2\sin^2 ax} + \frac{1}{2ab}\arcsin(b\sin ax)$

$(b^2 < 1)$

860. $\int \sin ax \sqrt{1+b^2\cos^2 ax}\,\mathrm{d}x = -\frac{\sqrt{1+b^2}\cos ax}{2a}\sqrt{1-\frac{b^2\sin^2 ax}{1+b^2}}$

$$-\frac{1}{ab}\ln\left(\sqrt{\frac{b^2\cos^2 ax}{1+b^2}} + \sqrt{1-\frac{b^2\sin^2 ax}{1+b^2}}\right)$$

861. $\int \cos ax \sqrt{1+b^2\cos^2 ax}\,\mathrm{d}x$

$$= \frac{\sqrt{1+b^2}\sin ax}{2a}\sqrt{1-\frac{b^2\sin^2 ax}{1+b^2}} + \frac{1+b^2}{2ab}\arcsin\sqrt{\frac{b^2\sin^2 ax}{1+b^2}}$$

862. $\int \sin ax \sqrt{1-b^2\cos^2 ax}\,\mathrm{d}x$

$$= -\frac{\sqrt{1-b^2}\cos ax}{2a}\sqrt{1+\frac{b^2\sin^2 ax}{1-b^2}} - \frac{\arcsin(b\cos ax)}{2ab} \quad (b^2 < 1)$$

863. $\int \cos ax \sqrt{1-b^2\cos^2 ax}\,\mathrm{d}x = \frac{\sqrt{1-b^2}\sin ax}{2a}\sqrt{1+\frac{b^2\sin^2 ax}{1-b^2}}$

$$-\frac{1-b^2}{2ab}\ln\left(\sqrt{\frac{b^2\sin^2 ax}{1-b^2}} + \sqrt{1+\frac{b^2\sin^2 ax}{1-b^2}}\right)$$

$(b^2 < 1)$

864. $\int \frac{\sin ax}{\sqrt{1+b^2\sin^2 ax}}\mathrm{d}x = -\frac{1}{ab}\arcsin\frac{b\cos ax}{\sqrt{1+b^2}}$

865. $\int \frac{\cos ax}{\sqrt{1+b^2\sin^2 ax}}\mathrm{d}x = \frac{1}{ab}\ln(b\sin ax + \sqrt{1+b^2\sin^2 ax})$

866. $\int \frac{\sin ax}{\sqrt{1-b^2\sin^2 ax}}\mathrm{d}x = -\frac{1}{ab}\ln(b\cos ax + \sqrt{1-b^2\sin^2 ax}) \quad (b^2 < 1)$

867. $\int \frac{\cos ax}{\sqrt{1-b^2\sin^2 ax}}\mathrm{d}x = \frac{1}{ab}\arcsin(b\sin ax) \quad (b^2<1)$

868. $\int \frac{\sin ax}{\sqrt{1+b^2\cos^2 ax}}\mathrm{d}x = -\frac{1}{ab}\ln\left(\sqrt{\frac{b^2\cos^2 ax}{1+b^2}}+\sqrt{1-\frac{b^2\sin^2 ax}{1+b^2}}\right)$

869. $\int \frac{\cos ax}{\sqrt{1+b^2\cos^2 ax}}\mathrm{d}x = \frac{1}{ab}\arcsin\sqrt{\frac{b^2\sin^2 ax}{1+b^2}}$

870. $\int \frac{\sin ax}{\sqrt{1-b^2\cos^2 ax}}\mathrm{d}x = -\frac{1}{ab}\arcsin(b\cos ax) \quad (b^2<1)$

871. $\int \frac{\cos ax}{\sqrt{1-b^2\cos^2 ax}}\mathrm{d}x = -\frac{1}{ab}\ln\left(\sqrt{\frac{b^2\sin^2 ax}{1-b^2}}+\sqrt{1+\frac{b^2\sin^2 ax}{1-b^2}}\right) \quad (b^2<1)$

Ⅰ.1.3.17 含有 $\sqrt{1-k^2\sin^2 x}$ 和 $\sqrt{a^2\sin^2 x-1}$ 的积分

在下列公式中，$0<k^2<1$.

872. $\int \sqrt{1-k^2\sin^2 x}\,\mathrm{d}x = \mathrm{E}(k,x)$

[这里，$\mathrm{E}(k,x)$ 为第二类椭圆积分(见附录)，以下同]

873. $\int \sin x\sqrt{1-k^2\sin^2 x}\,\mathrm{d}x$

$$= -\frac{1}{2}\cos x\sqrt{1-k^2\sin^2 x}-\frac{1-k^2}{2k}\ln(k\cos x+\sqrt{1-k^2\sin^2 x})$$

874. $\int \cos x\sqrt{1-k^2\sin^2 x}\,\mathrm{d}x = \frac{1}{2}\sin x\sqrt{1-k^2\sin^2 x}+\frac{1}{2k}\arcsin(k\sin x)$

875. $\int \frac{\sqrt{1-k^2\sin^2 x}}{\sin x}\mathrm{d}x$

$$= -\frac{1}{2}\ln\frac{\sqrt{1-k^2\sin^2 x}+\cos x}{\sqrt{1-k^2\sin^2 x}-\cos x}+k\ln(k\cos x+\sqrt{1-k^2\sin^2 x})$$

876. $\int \frac{\sqrt{1-k^2\sin^2 x}}{\cos x}\mathrm{d}x$

$$= \frac{\sqrt{1-k^2}}{2}\ln\frac{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}\sin x}{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}\sin x}+k\arcsin(k\sin x)$$

877. $\int \frac{\sin x\sqrt{1-k^2\sin^2 x}}{\cos x}\mathrm{d}x$

$$= \frac{\sqrt{1-k^2}}{2}\ln\frac{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}}-\sqrt{1-k^2\sin^2 x}$$

878. $\int \frac{\cos x\sqrt{1-k^2\sin^2 x}}{\sin x}\mathrm{d}x = \frac{1}{2}\ln\frac{1-\sqrt{1-k^2\sin^2 x}}{1+\sqrt{1-k^2\sin^2 x}}+\sqrt{1-k^2\sin^2 x}$

879. $\int \frac{\sqrt{1-k^2\sin^2 x}}{\sin^2 x}\mathrm{d}x = (1-k^2)\mathrm{F}(k,x) - \mathrm{E}(k,x) - \sqrt{1-k^2\sin^2 x}\cot x$

[这里，$\mathrm{F}(k,x)$ 为第一类椭圆积分(见附录)，以下同]

880. $\int \frac{\sqrt{1-k^2\sin^2 x}}{\cos^2 x}\mathrm{d}x = \mathrm{F}(k,x) - \mathrm{E}(k,x) + \sqrt{1-k^2\sin^2 x}\tan x$

881. $\int \frac{\sqrt{1-k^2\sin^2 x}}{\sin x\cos x}\mathrm{d}x$

$$= \frac{1}{2}\ln\frac{1-\sqrt{1-k^2\sin^2 x}}{1+\sqrt{1-k^2\sin^2 x}} + \frac{\sqrt{1-k^2}}{2}\ln\frac{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}}$$

882. $\int \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} = \mathrm{F}(k,x)$

883. $\int \frac{\mathrm{d}x}{\sin x\sqrt{1-k^2\sin^2 x}} = -\frac{1}{2}\ln\frac{\sqrt{1-k^2\sin^2 x}+\cos x}{\sqrt{1-k^2\sin^2 x}-\cos x}$

884. $\int \frac{\mathrm{d}x}{\cos x\sqrt{1-k^2\sin^2 x}} = -\frac{1}{2\sqrt{1-k^2}}\ln\frac{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}\sin x}{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}\sin x}$

885. $\int \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{2k}\ln\frac{\sqrt{1-k^2\sin^2 x}-k\cos x}{\sqrt{1-k^2\sin^2 x}+k\cos x}$

886. $\int \frac{\cos x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{k}\arcsin(k\sin x)$

887. $\int \frac{\sin x}{\cos x\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{2\sqrt{1-k^2}}\ln\frac{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}}$

888. $\int \frac{\cos x}{\sin x\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{2}\ln\frac{1-\sqrt{1-k^2\sin^2 x}}{1+\sqrt{1-k^2\sin^2 x}}$

889. $\int \frac{\sin^2 x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{k^2}\mathrm{F}(k,x) - \frac{1}{k^2}\mathrm{E}(k,x)$

890. $\int \frac{\cos^2 x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = -\frac{1-k^2}{k^2}\mathrm{F}(k,x) + \frac{1}{k^2}\mathrm{E}(k,x)$

891. $\int \frac{\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = -\frac{\sqrt{1-k^2\sin^2 x}}{k^2}$

892. $\int \frac{\mathrm{d}x}{\sin x\cos x\sqrt{1-k^2\sin^2 x}}$

$$= \frac{1}{2}\ln\frac{1-\sqrt{1-k^2\sin^2 x}}{1+\sqrt{1-k^2\sin^2 x}} + \frac{1}{2\sqrt{1-k^2}}\ln\frac{\sqrt{1-k^2\sin^2 x}+\sqrt{1-k^2}}{\sqrt{1-k^2\sin^2 x}-\sqrt{1-k^2}}$$

893. $\int \frac{\sin x}{\sqrt{a^2\sin^2 x-1}}\mathrm{d}x = -\frac{1}{a}\arcsin\frac{a\cos x}{\sqrt{a^2-1}} \quad (a^2>1)$

894. $\int \frac{\cos x}{\sqrt{a^2\sin^2 x-1}}\mathrm{d}x = \frac{1}{a}\ln\left|a\sin x+\sqrt{a^2\sin^2 x-1}\right| \quad (a^2>1)$

895. $\int \frac{dx}{\sin x\sqrt{a^2\sin^2 x-1}} = -\arctan\frac{\cos x}{\sqrt{a^2\sin^2 x-1}} \quad (a^2>1)$

896. $\int \frac{dx}{\cos x\sqrt{a^2\sin^2 x-1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}\sin x+\sqrt{a^2\sin^2 x-1}}{\sqrt{a^2-1}\sin x-\sqrt{a^2\sin^2 x-1}}$
$(a^2>1)$

897. $\int \frac{\tan x}{\sqrt{a^2\sin^2 x-1}}dx = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}+\sqrt{a^2\sin^2 x-1}}{\sqrt{a^2-1}-\sqrt{a^2\sin^2 x-1}} \quad (a^2>1)$

898. $\int \frac{\cot x}{\sqrt{a^2\sin^2 x-1}}dx = -\arcsin\frac{1}{a\sin x} \quad (a^2>1)$

Ⅰ.1.3.18 含有 $\tan ax$ 和 $\cot ax$ 的积分

899. $\int \frac{dx}{\tan ax} = \int \cot ax\,dx = \frac{1}{a}\ln|\sin ax|$

900. $\int \frac{dx}{\cot ax} = \int \tan ax\,dx = -\frac{1}{a}\ln|\cos ax|$

901. $\int x\tan ax\,dx = \frac{x}{a}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{4^k-1}{(2k+1)!}B_{2k}(2ax)^{2k} \quad \left(|ax|<\frac{\pi}{2}\right)$
[这里，B_{2k} 为伯努利数(见附录)，以下同]

902. $\int x\cot ax\,dx = \frac{x}{a}\left[1-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k+1)!}B_{2k}(2ax)^{2k}\right] \quad (|ax|<\pi)$

903. $\int \frac{\tan ax}{x}dx = \frac{1}{ax}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{4^k-1}{(2k)!(2k-1)}B_{2k}(2ax)^{2k} \quad \left(|ax|<\frac{\pi}{2}\right)$

904. $\int \frac{\cot ax}{x}dx = -\frac{1}{ax}\left[1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k+1)!}B_{2k}(2ax)^{2k}\right] \quad (|ax|<\pi)$

905. $\int \frac{dx}{p+q\tan ax} = \frac{1}{a(p^2+q^2)}(pax+q\ln|q\sin ax+p\cos ax|)$

906. $\int \frac{dx}{p+q\cot ax} = \frac{1}{a(p^2+q^2)}(pax-q\ln|p\sin ax+q\cos ax|)$

907. $\int \frac{\tan ax}{p+q\tan ax}dx = \frac{1}{a(p^2+q^2)}(qax-p\ln|q\sin ax+p\cos ax|)$

908. $\int \frac{\cot ax}{p+q\cot ax}dx = \frac{1}{a(p^2+q^2)}(qax+p\ln|p\sin ax+q\cos ax|)$

909. $\int \frac{\tan b-\tan ax}{\tan b+\tan ax}dx = \frac{1}{a}\sin 2b\ln|\sin(ax+b)|-x\cos 2b$

910. $\int \frac{\tan ax}{1+m^2\tan^2 ax}dx = \frac{\ln(\cos^2 ax+m^2\sin^2 ax)}{2a(m^2-1)}$

911. $\int x\tan^2 ax\,dx = \frac{x}{a}\tan ax+\frac{1}{a^2}\ln|\cos ax|-\frac{x^2}{2}$

912. $\int x\cot^2 ax\mathrm{d}x = -\frac{x}{a}\cot ax + \frac{1}{a^2}\ln|\sin ax| - \frac{x^2}{2}$

913. $\int \frac{\mathrm{d}x}{p+q\tan^2 ax} = \frac{x}{p-q} - \frac{1}{(p-q)a}\sqrt{\frac{q}{p}}\arctan\left(\sqrt{\frac{q}{p}}\tan ax\right) \quad (p \neq q)$

914. $\int \frac{\mathrm{d}x}{p+q\cot^2 ax} = \frac{x}{p-q} - \frac{1}{(p-q)a}\sqrt{\frac{q}{p}}\operatorname{arccot}\left(\sqrt{\frac{q}{p}}\cot ax\right) \quad (p \neq q)$

915. $\int \frac{\mathrm{d}x}{\sqrt{p+q\tan^2 ax}} = \frac{1}{a\sqrt{p-q}}\arcsin\left(\sqrt{\frac{p-q}{p}}\sin ax\right) \quad (p > q)$

916. $\int \frac{\mathrm{d}x}{\sqrt{p+q\cot^2 ax}} = \frac{1}{a\sqrt{p-q}}\arccos\left(\sqrt{\frac{p-q}{p}}\cos ax\right) \quad (p > q)$

917. $\int \frac{\sin ax}{\sqrt{p+q\tan^2 ax}}\mathrm{d}x = \frac{1}{a(q-p)}\sqrt{p\cos^2 ax + q\sin^2 ax} \quad (p \neq q)$

918. $\int \frac{\cos ax}{\sqrt{p+q\cot^2 ax}}\mathrm{d}x = \frac{1}{a\sqrt{p-q}}\sqrt{p\sin^2 ax + q\cos^2 ax} \quad (p > q)$

919. $\int \frac{\tan ax}{\sqrt{p+q\tan^2 ax}}\mathrm{d}x$

$= -\frac{1}{a\sqrt{p-q}}\ln\left|\sqrt{p-q}\cos ax + \sqrt{p\cos^2 ax + q\sin^2 ax}\right| \quad (p > q)$

920. $\int \frac{\cot ax}{\sqrt{p+q\cot^2 ax}}\mathrm{d}x = \frac{1}{a\sqrt{p-q}}\ln(\sqrt{p-q}\sin ax + \sqrt{p\sin^2 ax + q\cos^2 ax})$

$(p > q)$

Ⅰ.1.3.19 三角函数与代数函数组合的积分

921. $\int \frac{\sin kx}{a+bx}\mathrm{d}x = \frac{1}{b}\left\{\cos\frac{ka}{b}\mathrm{si}\left[\frac{k(a+bx)}{b}\right] - \sin\frac{ka}{b}\mathrm{ci}\left[\frac{k(a+bx)}{b}\right]\right\}$

[这里，si(x)和 ci(x)分别为正弦积分和余弦积分(见附录)，以下同]

922. $\int \frac{\cos kx}{a+bx}\mathrm{d}x = \frac{1}{b}\left\{\cos\frac{ka}{b}\mathrm{ci}\left[\frac{k(a+bx)}{b}\right] + \sin\frac{ka}{b}\mathrm{si}\left[\frac{k(a+bx)}{b}\right]\right\}$

923. $\int \frac{\sin kx}{(a+bx)^2}\mathrm{d}x = -\frac{\sin kx}{b(a+bx)} + \frac{k}{b}\int\frac{\cos kx}{a+bx}\mathrm{d}x$

924. $\int \frac{\cos kx}{(a+bx)^2}\mathrm{d}x = -\frac{\cos kx}{b(a+bx)} - \frac{k}{b}\int\frac{\sin kx}{a+bx}\mathrm{d}x$

925. $\int \frac{\sin x}{\sqrt{x}}\mathrm{d}x = \sqrt{2\pi}\mathrm{S}(\sqrt{x})$

[这里，S(x)为菲涅耳积分(见附录)]

926. $\int \frac{\cos x}{\sqrt{x}}\mathrm{d}x = \sqrt{2\pi}\mathrm{C}(\sqrt{x})$

[这里,C(x)为菲涅耳积分(见附录)]

Ⅰ.1.3.20 三角函数与指数函数和双曲函数组合的积分

927. $\int e^{ax}\sin^{2n}bx\,dx$

$$= \binom{2n}{n}\frac{e^{ax}}{2^{2n}a} + \frac{e^{ax}}{2^{2n-1}}\sum_{k=1}^{m}\binom{2n}{n-k}\frac{(-1)^k}{a^2+4b^2k^2}(a\cos 2bkx + 2bk\sin 2bkx)$$

928. $\int e^{ax}\sin^{2n+1}bx\,dx = \frac{e^{ax}}{2^{2n}}\sum_{k=0}^{n}\left\{\binom{2n+1}{n-k}\frac{(-1)^k}{a^2+(2k+1)^2b^2}\right.$

$$\left.\cdot\left[a\sin(2k+1)bx - (2k+1)b\cos(2k+1)bx\right]\right\}$$

929. $\int e^{ax}\cos^{2n}bx\,dx$

$$= \binom{2n}{n}\frac{e^{ax}}{2^{2n}a} + \frac{e^{ax}}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n-k}\frac{1}{a^2+4b^2k^2}(a\cos 2bkx + 2bk\sin 2bkx)$$

930. $\int e^{ax}\cos^{2n+1}bx\,dx = \frac{e^{ax}}{2^{2n}}\sum_{k=0}^{n}\left\{\binom{2n+1}{n-k}\frac{1}{a^2+(2k+1)^2b^2}\right.$

$$\left.\cdot\left[a\cos(2k+1)bx + (2k+1)b\sin(2k+1)bx\right]\right\}$$

931. $\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}}{2}\left[\frac{a\sin(b+c)x-(b+c)\cos(b+c)x}{a^2+(b+c)^2}\right.$

$$\left.+\frac{a\sin(b-c)x-(b-c)\cos(b-c)x}{a^2+(b-c)^2}\right]$$

932. $\int xe^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax-\frac{a^2-b^2}{a^2+b^2}\right)\sin bx - \left(bx-\frac{2ab}{a^2+b^2}\right)\cos bx\right]$

933. $\int xe^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax-\frac{a^2-b^2}{a^2+b^2}\right)\cos bx + \left(bx-\frac{2ab}{a^2+b^2}\right)\sin bx\right]$

934. $\int x^m e^{ax}\sin bx\,dx = e^{ax}\sum_{k=1}^{m+1}\frac{(-1)^{k+1}m!\,x^{m-k+1}}{(m-k+1)!(a^2+b^2)^{\frac{k}{2}}}\sin(bx+kt)$

$\left(\text{这里},\sin t = -\frac{b}{\sqrt{a^2+b^2}}\right)$

935. $\int x^m e^{ax}\cos bx\,dx = e^{ax}\sum_{k=1}^{m+1}\frac{(-1)^{k+1}m!\,x^{m-k+1}}{(m-k+1)!(a^2+b^2)^{\frac{k}{2}}}\cos(bx+kt)$

$\left(\text{这里},\cos t = \frac{a}{\sqrt{a^2+b^2}}\right)$

936. $\int \sinh(ax+b)\sin(cx+d)\,dx$

$$= \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d)$$

937. $\int \sinh(ax+b)\cos(cx+d)\mathrm{d}x$

$$= \frac{a}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\sin(cx+d)$$

938. $\int \cosh(ax+b)\sin(cx+d)\mathrm{d}x$

$$= \frac{a}{a^2+c^2}\sinh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\cos(cx+d)$$

939. $\int \cosh(ax+b)\cos(cx+d)\mathrm{d}x$

$$= \frac{a}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\sin(cx+d)$$

Ⅰ.1.3.21 含有 $\sin x^2$,$\cos x^2$ 和更复杂自变数的三角函数的积分

940. $\int \sin x^2 \mathrm{d}x = \sqrt{\frac{\pi}{2}}\mathrm{S}(x)$

[这里,$\mathrm{S}(x)$为菲涅耳积分(见附录),以下同]

941. $\int \cos x^2 \mathrm{d}x = \sqrt{\frac{\pi}{2}}\mathrm{C}(x)$

[这里,$\mathrm{C}(x)$为菲涅耳积分(见附录),以下同]

942. $\int x\sin x^2 \mathrm{d}x = -\frac{\cos x^2}{2}$

943. $\int x\cos x^2 \mathrm{d}x = \frac{\sin x^2}{2}$

944. $\int x^2\sin x^2 \mathrm{d}x = -\frac{x}{2}\cos x^2 + \frac{1}{2}\sqrt{\frac{\pi}{2}}\mathrm{C}(x)$

945. $\int x^2\cos x^2 \mathrm{d}x = \frac{x}{2}\sin x^2 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\mathrm{S}(x)$

946. $\int x^3\sin x^2 \mathrm{d}x = \frac{1}{2}\sin x^2 - \frac{x^2}{2}\cos x^2$

947. $\int x^3\cos x^2 \mathrm{d}x = \frac{1}{2}\cos x^2 + \frac{x^2}{2}\sin x^2$

948. $\int \sin(ax^2+2bx+c)\mathrm{d}x$

$$= \sqrt{\frac{\pi}{2a}}\left[\cos\frac{ac-b^2}{a}\mathrm{S}\left(\frac{ax+b}{\sqrt{a}}\right) + \sin\frac{ac-b^2}{a}\mathrm{C}\left(\frac{ax+b}{\sqrt{a}}\right)\right]$$

949. $\int \cos(ax^2+2bx+c)\mathrm{d}x$

$$= \sqrt{\frac{\pi}{2a}}\left[\cos\frac{ac-b^2}{a}C\left(\frac{ax+b}{\sqrt{a}}\right) - \sin\frac{ac-b^2}{a}S\left(\frac{ax+b}{\sqrt{a}}\right)\right]$$

950. $\int \sin(\ln x)\mathrm{d}x = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)]$

951. $\int \cos(\ln x)\mathrm{d}x = \frac{x}{2}[\sin(\ln x) + \cos(\ln x)]$

952. $\int x^p \sin(b\ln x)\mathrm{d}x = \frac{x^{p+1}}{(p+1)^2+b^2}[(p+1)\sin(b\ln x) - b\cos(b\ln x)]$

953. $\int x^p \cos(b\ln x)\mathrm{d}x = \frac{x^{p+1}}{(p+1)^2+b^2}[(p+1)\cos(b\ln x) + b\sin(b\ln x)]$

Ⅰ.1.3.22 反三角函数的积分

954. $\int \arcsin ax\mathrm{d}x = x\arcsin ax + \frac{\sqrt{1-a^2x^2}}{a}$

955. $\int \arccos ax\mathrm{d}x = x\arccos ax - \frac{\sqrt{1-a^2x^2}}{a}$

956. $\int \arctan ax\mathrm{d}x = x\arctan ax - \frac{1}{2a}\ln(1+a^2x^2)$

957. $\int \operatorname{arccot} ax\mathrm{d}x = x\operatorname{arccot} ax + \frac{1}{2a}\ln(1+a^2x^2)$

958. $\int \operatorname{arcsec} ax\mathrm{d}x = x\operatorname{arcsec} ax - \frac{1}{a}\ln|ax+\sqrt{a^2x^2-1}|$

959. $\int \operatorname{arccsc} ax\mathrm{d}x = x\operatorname{arccsc} ax + \frac{1}{a}\ln|ax+\sqrt{a^2x^2-1}|$

960. $\int x\arcsin ax\mathrm{d}x = \frac{1}{4a^2}[(2a^2x^2-1)\arcsin ax + ax\sqrt{1-a^2x^2}]$

961. $\int x\arccos ax\mathrm{d}x = \frac{1}{4a^2}[(2a^2x^2-1)\arccos ax - ax\sqrt{1-a^2x^2}]$

962. $\int x\arctan ax\mathrm{d}x = \frac{1+a^2x^2}{2a^2}\arctan ax - \frac{x}{2a}$

963. $\int x\operatorname{arccot} ax\mathrm{d}x = \frac{1+a^2x^2}{2a^2}\operatorname{arccot} ax + \frac{x}{2a}$

964. $\int x\operatorname{arcsec} ax\mathrm{d}x = \frac{x^2}{2}\operatorname{arcsec} ax - \frac{1}{2a^2}\sqrt{a^2x^2-1}$

965. $\int x\operatorname{arccsc} ax\mathrm{d}x = \frac{x^2}{2}\operatorname{arccsc} ax + \frac{1}{2a^2}\sqrt{a^2x^2-1}$

966. $\int x^n\arcsin ax\mathrm{d}x = \frac{x^{n+1}}{n+1}\arcsin ax - \frac{a}{n+1}\int\frac{x^{n+1}}{\sqrt{1-a^2x^2}}\mathrm{d}x \quad (n\neq -1)$

967. $\int x^n \arccos ax \mathrm{d}x = \frac{x^{n+1}}{n+1}\arccos ax + \frac{a}{n+1}\int \frac{x^{n+1}}{\sqrt{1-a^2x^2}}\mathrm{d}x \quad (n \neq -1)$

968. $\int x^n \arctan ax \mathrm{d}x = \frac{x^{n+1}}{n+1}\arctan ax - \frac{a}{n+1}\int \frac{x^{n+1}}{1+a^2x^2}\mathrm{d}x \quad (n \neq -1)$

969. $\int x^n \operatorname{arccot} ax \mathrm{d}x = \frac{x^{n+1}}{n+1}\operatorname{arccot} ax + \frac{a}{n+1}\int \frac{x^{n+1}}{1+a^2x^2}\mathrm{d}x \quad (n \neq -1)$

970. $\int x^n \operatorname{arcsec} ax \mathrm{d}x = \frac{x^{n+1}}{n+1}\operatorname{arcsec} ax - \frac{1}{n+1}\int \frac{x^n}{\sqrt{a^2x^2-1}}\mathrm{d}x \quad (n \neq -1)$

971. $\int x^n \operatorname{arccsc} ax \mathrm{d}x = \frac{x^{n+1}}{n+1}\operatorname{arccsc} ax + \frac{1}{n+1}\int \frac{x^n}{\sqrt{a^2x^2-1}}\mathrm{d}x \quad (n \neq -1)$

972. $\int (\arcsin ax)^2 \mathrm{d}x = x(\arcsin ax)^2 - 2x + \frac{2\sqrt{1-a^2x^2}}{a}\arcsin ax$

973. $\int (\arccos ax)^2 \mathrm{d}x = x(\arccos ax)^2 - 2x - \frac{2\sqrt{1-a^2x^2}}{a}\arccos ax$

974. $$\begin{aligned}\int (\arcsin ax)^n \mathrm{d}x &= x(\arcsin ax)^n + \frac{n\sqrt{1-a^2x^2}}{a}(\arcsin ax)^{n-1} \\ &\quad - n(n-1)\int (\arcsin ax)^{n-2}\mathrm{d}x \\ &= \sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r \frac{n!}{(n-2r)!}x(\arcsin ax)^{n-2r} \\ &\quad + \sum_{r=0}^{\left[\frac{n-1}{2}\right]}(-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a}(\arcsin ax)^{n-2r-1}\end{aligned}$$

975. $$\begin{aligned}\int (\arccos ax)^n \mathrm{d}x &= x(\arccos ax)^n - \frac{n\sqrt{1-a^2x^2}}{a}(\arccos ax)^{n-1} \\ &\quad - n(n-1)\int (\arccos ax)^{n-2}\mathrm{d}x \\ &= \sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r \frac{n!}{(n-2r)!}x(\arccos ax)^{n-2r} \\ &\quad - \sum_{r=0}^{\left[\frac{n-1}{2}\right]}(-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a}(\arccos ax)^{n-2r-1}\end{aligned}$$

976. $$\begin{aligned}&\int \frac{\arcsin ax}{x}\mathrm{d}x \\ &= ax + \frac{1}{2\cdot 3\cdot 3}(ax)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}(ax)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}(ax)^7 + \cdots \\ &= \sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!(2k+1)^2}(ax)^{2k+1} \quad (|ax|<1)\end{aligned}$$

977. $\int \frac{\arccos ax}{x}\mathrm{d}x = \frac{\pi}{2}\ln|x| - ax - \frac{1}{2\cdot 3\cdot 3}(ax)^3 - \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}(ax)^5$

$$-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot7}(ax)^7-\cdots$$

$$=\frac{\pi}{2}\ln|x|-\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!(2k+1)^2}(ax)^{2k+1}\quad(|ax|<1)$$

978. $\int\frac{\arctan ax}{x}\mathrm{d}x=\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^2}(ax)^{2k+1}\quad(|ax|<1)$

979. $\int\frac{\operatorname{arccot} ax}{x}\mathrm{d}x=\frac{\pi}{2}\ln|x|-\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^2}(ax)^{2k+1}\quad(|ax|<1)$

980. $\int\frac{\arcsin ax}{x^2}\mathrm{d}x=-\frac{1}{x}\arcsin ax+a\ln\left|\frac{1-\sqrt{1-a^2x^2}}{x}\right|$

981. $\int\frac{\arccos ax}{x^2}\mathrm{d}x=-\frac{1}{x}\arccos ax+a\ln\left|\frac{1+\sqrt{1-a^2x^2}}{x}\right|$

982. $\int\frac{\arctan ax}{x^2}\mathrm{d}x=-\frac{1}{x}\arctan ax-\frac{a}{2}\ln\frac{1+a^2x^2}{x^2}$

983. $\int\frac{\operatorname{arccot} ax}{x^2}\mathrm{d}x=-\frac{1}{x}\operatorname{arccot} ax-\frac{a}{2}\ln\frac{x^2}{1+a^2x^2}$

984. $\int\frac{\operatorname{arcsec} ax}{x^2}\mathrm{d}x=-\frac{1}{x}\operatorname{arcsec} ax+\frac{\sqrt{a^2x^2-1}}{x}$

985. $\int\frac{\operatorname{arccsc} ax}{x^2}\mathrm{d}x=-\frac{1}{x}\operatorname{arccsc} ax-\frac{\sqrt{a^2x^2-1}}{x}$

986. $\int\frac{\arcsin ax}{\sqrt{1-a^2x^2}}\mathrm{d}x=\frac{1}{2a}(\arcsin ax)^2$

987. $\int\frac{\arccos ax}{\sqrt{1-a^2x^2}}\mathrm{d}x=-\frac{1}{2a}(\arccos ax)^2$

988. $\int\frac{x^n\arcsin ax}{\sqrt{1-a^2x^2}}\mathrm{d}x=-\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\arcsin ax+\frac{x^n}{n^2a}+\frac{n-1}{na^2}\int\frac{x^{n-2}\arcsin ax}{\sqrt{1-a^2x^2}}\mathrm{d}x$

989. $\int\frac{x^n\arccos ax}{\sqrt{1-a^2x^2}}\mathrm{d}x=-\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\arccos ax-\frac{x^n}{n^2a}+\frac{n-1}{na^2}\int\frac{x^{n-2}\arccos ax}{\sqrt{1-a^2x^2}}\mathrm{d}x$

990. $\int\frac{\arctan ax}{1+a^2x^2}\mathrm{d}x=\frac{1}{2a}(\arctan ax)^2$

991. $\int\frac{\operatorname{arccot} ax}{1+a^2x^2}\mathrm{d}x=-\frac{1}{2a}(\operatorname{arccot} ax)^2$

Ⅰ.1.4 对数函数、指数函数和双曲函数的不定积分

Ⅰ.1.4.1 对数函数的积分

992. $\int \ln x \, dx = x \ln x - x$

993. $\int (\ln x)^2 dx = x(\ln x)^2 - 2x \ln x + 2x$

994. $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx \quad (n \neq -1)$

$$= (-1)^n n! x \sum_{r=0}^{n} \frac{(-\ln x)^r}{r!} \quad (n \neq -1)$$

995. $\int x \ln x \, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$

996. $\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9}$

997. $\int x^m \ln x \, dx = \frac{x^{m+1}}{m+1}\ln x - \frac{x^{m+1}}{(m+1)^2} \quad (m \neq -1)$

998. $\int x^m (\ln x)^2 dx = x^{m+1}\left[\frac{(\ln x)^2}{m+1} - \frac{2\ln x}{(m+1)^2} + \frac{2}{(m+1)^3}\right] \quad (m \neq -1)$

999. $\int x^m (\ln x)^3 dx = x^{m+1}\left[\frac{(\ln x)^3}{m+1} - \frac{3(\ln x)^2}{(m+1)^2} + \frac{6\ln x}{(m+1)^3} - \frac{6}{(m+1)^4}\right]$
$(m \neq -1)$

1000. $\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx \quad (m \neq -1)$

$$= (-1)^n \frac{n!}{m+1} x^{m+1} \sum_{r=0}^{n} \frac{(-\ln x)^r}{r!(m+1)^{n-r}} \quad (m \neq -1)$$

1001. $\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2$

1002. $\int \frac{\ln x}{x^m} dx = -\frac{1 + (m-1)\ln x}{(m-1)^2 x^{m-1}} \quad (m \neq 1)$

1003. $\int \frac{(\ln x)^n}{x} dx = \frac{1}{n+1}(\ln x)^{n+1} \quad (n \neq -1)$

1004. $\int \frac{(\ln x)^n}{x^m} dx = -\frac{(\ln x)^n}{(m-1)x^{m-1}} + \frac{n}{m-1}\int \frac{(\ln x)^{n-1}}{x^m} dx \quad (m \neq 1)$

1005. $\int \frac{dx}{\ln x} = \ln(\ln x) + \ln x + \frac{(\ln x)^2}{2 \cdot 2!} + \frac{(\ln x)^3}{3 \cdot 3!} + \cdots$

1006. $\int \frac{x^m}{\ln x}\mathrm{d}x$

$$= \ln(\ln x) + (m+1)\ln x + \frac{(m+1)^2(\ln x)^2}{2\cdot 2!} + \frac{(m+1)^3(\ln x)^3}{3\cdot 3!} + \cdots$$

1007. $\int \frac{x^m}{(\ln x)^n}\mathrm{d}x = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m}{(\ln x)^{n-1}}\mathrm{d}x \quad (n \neq 1)$

1008. $\int \frac{\mathrm{d}x}{x\ln x} = \ln(\ln x)$

1009. $\int \frac{\mathrm{d}x}{x(\ln x)^n} = -\frac{1}{(n-1)(\ln x)^{n-1}} \quad (n \neq 1)$

1010. $\int \frac{\mathrm{d}x}{x^m(\ln x)^n} = -\frac{1}{(n-1)x^{m-1}(\ln x)^{n-1}} - \frac{(m-1)}{(n-1)}\int \frac{\mathrm{d}x}{x^m(\ln x)^{n-1}} \quad (n \neq 1)$

1011. $\int \frac{\ln x}{ax+b}\mathrm{d}x = \frac{1}{a}\ln x \ln(ax+b) - \frac{1}{a}\int \frac{\ln(ax+b)}{x}\mathrm{d}x$

1012. $\int \frac{\ln x}{(ax+b)^2}\mathrm{d}x = -\frac{\ln x}{a(ax+b)} + \frac{1}{ab}\ln\left|\frac{x}{ax+b}\right|$

1013. $\int \frac{\ln x}{(ax+b)^3}\mathrm{d}x = -\frac{\ln x}{2a(ax+b)^2} + \frac{1}{2ab(ax+b)} + \frac{1}{2ab^2}\ln\left|\frac{x}{ax+b}\right|$

1014. $\int \frac{\ln x}{(ax+b)^m}\mathrm{d}x = \frac{1}{b(m-1)}\left[-\frac{\ln x}{(ax+b)^{m-1}} + \int \frac{\mathrm{d}x}{x(ax+b)^{m-1}}\right]$

1015. $\int \frac{\ln x}{\sqrt{ax+b}}\mathrm{d}x$

$$= \begin{cases} \frac{2}{a}\left[(\ln x - 2)\sqrt{ax+b} + \sqrt{b}\ln\left|\frac{\sqrt{ax+b}+\sqrt{b}}{\sqrt{ax+b}-\sqrt{b}}\right|\right] & (b > 0) \\ \frac{2}{a}\left[(\ln x - 2)\sqrt{ax+b} + 2\sqrt{-b}\arctan\sqrt{\frac{ax+b}{-b}}\right] & (b < 0) \end{cases}$$

1016. $\int \ln(ax+b)\mathrm{d}x = \frac{ax+b}{a}\ln(ax+b) - x$

1017. $\int x\ln(ax+b)\mathrm{d}x = \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right)\ln(ax+b) - \frac{1}{2}\left(\frac{x^2}{2} - \frac{bx}{a}\right)$

1018. $\int x^2\ln(ax+b)\mathrm{d}x = \frac{1}{3}\left(x^3 + \frac{b^3}{a^3}\right)\ln(ax+b) - \frac{1}{3}\left(\frac{x^3}{3} - \frac{bx^2}{2a} + \frac{b^2x}{a^2}\right)$

1019. $\int x^3\ln(ax+b)\mathrm{d}x$

$$= \frac{1}{4}\left(x^4 - \frac{b^4}{a^4}\right)\ln(ax+b) - \frac{1}{4}\left(\frac{x^4}{4} - \frac{bx^3}{3a} + \frac{b^2x^2}{2a^2} - \frac{b^3x}{a^3}\right)$$

1020. $\int x^m\ln(ax+b)\mathrm{d}x = \frac{1}{m+1}\left[x^{m+1} - \left(-\frac{b}{a}\right)^{m+1}\right]\ln(ax+b)$

$$- \frac{1}{m+1}\left(-\frac{b}{a}\right)^{m+1}\sum_{r=1}^{m+1}\frac{1}{r}\left(-\frac{ax}{b}\right)^r$$

1021. $\int \frac{\ln(ax+b)}{x}\mathrm{d}x = \ln b \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}\left(\frac{ax}{b}\right)^k \quad (-b < ax \leqslant b)$

1022. $\int \frac{\ln(ax+b)}{x^2}\mathrm{d}x = \frac{a}{b}\ln x - \frac{ax+b}{bx}\ln(ax+b)$

1023. $\int \frac{\ln(ax+b)}{x^m}\mathrm{d}x = -\frac{1}{m-1}\frac{\ln(ax+b)}{x^{m-1}} + \frac{1}{m-1}\left(-\frac{b}{a}\right)^{m-1}\ln\frac{ax+b}{x}$

$$+\frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\sum_{r=1}^{m-2}\frac{1}{r}\left(-\frac{b}{ax}\right)^r \quad (m>2)$$

1024. $\int \ln\frac{x+a}{x-a}\mathrm{d}x = (x+a)\ln(x+a) - (x-a)\ln(x-a)$

1025. $\int x^m \ln\frac{x+a}{x-a}\mathrm{d}x = \frac{x^{m+1}-(-a)^{m+1}}{m+1}\ln(x+a) - \frac{x^{m+1}-a^{m+1}}{m+1}\ln(x-a)$

$$+\frac{2a^{m+1}}{m+1}\sum_{r=1}^{\left[\frac{m+1}{2}\right]}\frac{1}{m-2r+2}\left(\frac{x}{a}\right)^{m-2r+2} \quad (m \neq -1)$$

1026. $\int \frac{1}{x^2}\ln\frac{x+a}{x-a}\mathrm{d}x = \frac{1}{x}\ln\frac{x-a}{x+a} - \frac{1}{a}\ln\frac{x^2-a^2}{x^2}$

1027. $\int \ln(x^2+a^2)\mathrm{d}x = x\ln(x^2+a^2) - 2x + 2a\arctan\frac{x}{a}$

1028. $\int x\ln(x^2+a^2)\mathrm{d}x = \frac{1}{2}(x^2+a^2)\ln(x^2+a^2) - \frac{1}{2}x^2$

1029. $\int x^2\ln(x^2+a^2)\mathrm{d}x = \frac{1}{3}\left[x^3\ln(x^2+a^2) - \frac{2}{3}x^3 + 2a^2x - 2a^3\arctan\frac{x}{a}\right]$

1030. $\int x^{2n}\ln(x^2+a^2)\mathrm{d}x = \frac{1}{2n+1}\left[x^{2n+1}\ln(x^2+a^2) + (-1)^n 2a^{2n+1}\arctan\frac{x}{a}\right.$

$$\left.-2\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2k+1}a^{2n-2k}x^{2k+1}\right]$$

1031. $\int x^{2n+1}\ln(x^2+a^2)\mathrm{d}x$

$$=\frac{1}{2n+1}\left\{\left[x^{2n+2}+(-1)^n a^{2n+2}\right]\ln(x^2+a^2) + \sum_{k=1}^{n+1}\frac{(-1)^{n-k}}{k}a^{2n-2k+2}x^{2k}\right\}$$

1032. $\int \ln|x^2-a^2|\mathrm{d}x = x\ln|x^2-a^2| - 2x + a\ln\left|\frac{x+a}{x-a}\right|$

1033. $\int x\ln|x^2-a^2|\mathrm{d}x = \frac{1}{2}\left[(x^2-a^2)\ln|x^2-a^2| - x^2\right]$

1034. $\int x^2\ln|x^2-a^2|\mathrm{d}x = \frac{1}{3}\left(x^3\ln|x^2-a^2| - \frac{2}{3}x^3 - 2a^2x + a^3\ln\left|\frac{x+a}{x-a}\right|\right)$

1035. $\int x^{2n}\ln|x^2-a^2|\mathrm{d}x$

$$=\frac{1}{2n+1}\left(x^{2n+1}\ln|x^2-a^2| + a^{2n+1}\ln\left|\frac{x+a}{x-a}\right| - 2\sum_{k=0}^{n}\frac{1}{2k+1}a^{2n-2k}x^{2k+1}\right)$$

1036. $\int x^{2n+1}\ln|x^2-a^2|\mathrm{d}x$

$$=\frac{1}{2n+2}\left[(x^{2n+2}-a^{2n+2})\ln|x^2-a^2|-\sum_{k=1}^{n+1}\frac{1}{k}a^{2n-2k+2}x^{2k}\right]$$

1037. $\int\ln(x+\sqrt{x^2\pm a^2})\mathrm{d}x=x\ln(x+\sqrt{x^2\pm a^2})-\sqrt{x^2\pm a^2}$

1038. $\int x\ln(x+\sqrt{x^2\pm a^2})\mathrm{d}x=\left(\frac{x^2}{2}\pm\frac{a^2}{4}\right)\ln(x+\sqrt{x^2\pm a^2})-\frac{x\sqrt{x^2\pm a^2}}{4}$

1039. $\int x^m\ln(x+\sqrt{x^2\pm a^2})\mathrm{d}x=\frac{x^{m+1}}{m+1}\ln(x+\sqrt{x^2\pm a^2})-\frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2\pm a^2}}\mathrm{d}x$

1040. $\int\frac{\ln(x+\sqrt{x^2+a^2})}{x^2}\mathrm{d}x=-\frac{\ln(x+\sqrt{x^2+a^2})}{x}-\frac{1}{a}\ln\frac{a+\sqrt{x^2+a^2}}{x}$

1041. $\int\frac{\ln(x+\sqrt{x^2-a^2})}{x^2}\mathrm{d}x=-\frac{\ln(x+\sqrt{x^2-a^2})}{x}-\frac{1}{|a|}\operatorname{arcsec}\frac{x}{a}$

1042. $\int\ln X\mathrm{d}x$

$$=\begin{cases}\left(x+\frac{b}{2c}\right)\ln X-2x+\frac{\sqrt{4ac-b^2}}{c}\arctan\frac{2cx+b}{\sqrt{4ac-b^2}} & (b^2-4ac<0)\\ \left(x+\frac{b}{2c}\right)\ln X-2x+\frac{\sqrt{b^2-4ac}}{c}\operatorname{artanh}\frac{2cx+b}{\sqrt{b^2-4ac}} & (b^2-4ac>0)\end{cases}$$

(这里，$X=a+bx+cx^2$)

1043. $\int x^n\ln X\mathrm{d}x=\frac{x^{n+1}}{n+1}\ln X-\frac{2c}{n+1}\int\frac{x^{n+2}}{X}\mathrm{d}x-\frac{b}{n+1}\int\frac{x^{n+1}}{X}\mathrm{d}x\quad(n\neq-1)$

(这里，$X=a+bx+cx^2$)

1044. $\int\ln(\sin x)\mathrm{d}x=x\left[\ln x-1-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}B_{2k}(2x)^{2k}}{2k(2k)!(2k+1)}\right]\quad(|x|<\pi)$

[这里，B_{2k}为伯努利数(见附录)，以下同]

1045. $\int\ln(\cos x)\mathrm{d}x=x\sum_{k=1}^{\infty}(-1)^k\frac{(4^k-1)B_{2k}(2x)^{2k}}{2k(2k)!(2k+1)}\quad\left(|x|<\frac{\pi}{2}\right)$

Ⅰ.1.4.2 指数函数的积分

1046. $\int\mathrm{e}^x\mathrm{d}x=\mathrm{e}^x$

1047. $\int\mathrm{e}^{-x}\mathrm{d}x=-\mathrm{e}^{-x}$

1048. $\int\mathrm{e}^{ax}\mathrm{d}x=\frac{1}{a}\mathrm{e}^{ax}$

1049. $\int x\,e^{ax}dx = \frac{1}{a^2}e^{ax}(ax-1)$

1050. $\int x^2e^{ax}dx = \frac{x^2e^{ax}}{a}\left[1-\frac{2}{ax}\left(1-\frac{1}{ax}\right)\right]$

1051. $\int x^m e^{ax}dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}dx = e^{ax}\sum_{r=0}^{m}(-1)^r\frac{m!\,x^{m-r}}{(m-r)!\,a^{r+1}}$

1052. $\int \frac{e^{ax}}{\sqrt{x}}dx = 2\sqrt{x}\left(1+\frac{ax}{1\cdot 3}\left\{1+\frac{3ax}{2\cdot 5}\left[1+\frac{5ax}{3\cdot 7}(1+\cdots)\right]\right\}\right)$

1053. $\int \frac{e^{ax}}{x}dx = \ln x + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \cdots = \mathrm{Ei}(ax)$

[这里,$\mathrm{Ei}(ax)$ 为指数积分(见附录)]

1054. $\int \frac{e^{ax}}{x^2}dx = -\frac{e^{ax}}{x} + \int \frac{e^{ax}}{x}dx$

1055. $\int \frac{e^{ax}}{x^m}dx = \frac{1}{1-m}\frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1}\int \frac{e^{ax}}{x^{m-1}}dx \quad (m \neq 1)$

1056. $\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}dx$

1057. $\int a^{px}dx = \frac{a^{px}}{p\ln a}$

(这里,积分式中的 a^{px} 可用 $a^{px} = e^{px\ln a}$ 代替,以下同)

1058. $\int x\,a^{px}dx = \frac{(px\ln a-1)a^{px}}{(p\ln a)^2}$

1059. $\int x^2a^{px}dx = \frac{x^3a^{px}}{px\ln a}\left[1-\frac{2}{px\ln a}\left(1-\frac{1}{px\ln a}\right)\right]$

1060. $\int x^m a^{px}dx = \frac{x^{m+1}a^{px}}{px\ln a}\left(1-\frac{m}{px\ln a}\left\{1-\frac{m-1}{px\ln a}\left[1-\frac{m-2}{px\ln a}(1-\cdots)\right]\right\}\right)$

1061. $\int \frac{a^{px}}{\sqrt{x}}dx = 2\sqrt{x}\left(1+\frac{px\ln a}{1\cdot 3}\left\{1+\frac{3px\ln a}{2\cdot 5}\left[1+\frac{5px\ln a}{3\cdot 7}(1+\cdots)\right]\right\}\right)$

1062. $\int \frac{a^{px}}{x}dx = \ln x + \frac{px\ln a}{1\cdot 1}\left(1+\frac{px\ln a}{2\cdot 2}\left\{1+\frac{2px\ln a}{3\cdot 3}\left[1+\frac{3px\ln a}{4\cdot 4}(1+\cdots)\right]\right\}\right)$

1063. $\int \frac{a^{px}}{x^2}dx = -\frac{a^{px}}{x} + p\ln a\int \frac{a^{px}}{x}dx$

1064. $\int \frac{a^{px}}{x^m}dx = -\frac{a^{px}}{(m-1)x^{m-1}}\left\{1+\frac{px\ln a}{m-2}\left[1+\frac{px\ln a}{m-3}(1+\cdots+px\ln a)\right]\right\}$

$+\frac{(p\ln a)^{m-1}}{(m-1)!}\int \frac{a^{px}}{x}dx$

1065. $\int (a^x - a^{-x})dx = \frac{a^x + a^{-x}}{\ln a}$

1066. $\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\ln(a+be^{px})$

1067. $\int \frac{dx}{\sqrt{a+be^{px}}} = \begin{cases} \frac{1}{p\sqrt{a}}\ln\frac{\sqrt{a+be^{px}}-\sqrt{a}}{\sqrt{a+be^{px}}+\sqrt{a}} & (a>0,\ b>0) \\ \frac{1}{p\sqrt{-a}}\arctan\frac{\sqrt{a+be^{px}}}{\sqrt{-a}} & (a<0,\ b>0) \end{cases}$

1068. $\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}}\arctan\left(e^{mx}\sqrt{\frac{a}{b}}\right) \quad (a>0,\ b>0)$

1069. $\int \frac{dx}{ae^{mx}-be^{-mx}} = \frac{1}{2m\sqrt{ab}}\ln\left|\frac{\sqrt{a}e^{mx}-\sqrt{b}}{\sqrt{a}e^{mx}+\sqrt{b}}\right| \quad (a>0,\ b>0)$

$= \frac{1}{m\sqrt{ab}}\text{artanh}\left(\sqrt{\frac{a}{b}}e^{mx}\right) \quad (a>0,\ b>0)$

1070. $\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}$

1071. $\int \frac{e^{ax}}{b+ce^{ax}}dx = \frac{1}{ac}\ln(b+ce^{ax})$

1072. $\int \frac{xe^{ax}}{(1+ax)^2}dx = \frac{e^{ax}}{a^2(1+ax)}$

1073. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

1074. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

1075. $\int e^{ax}\sin bx\sin cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b+c)^2]} - \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$

1076. $\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}[a\sin(b-c)x - (b-c)\cos(b-c)x]}{2[a^2+(b-c)^2]} + \frac{e^{ax}[a\sin(b+c)x - (b+c)\cos(b+c)x]}{2[a^2+(b+c)^2]}$

1077. $\int e^{ax}\cos bx\cos cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]} + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$

1078. $\int e^{ax}\sin bx\sin(bx+c)dx = \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$

1079. $\int e^{ax}\sin bx\cos(bx+c)dx = -\frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$

1080. $\int e^{ax}\cos bx\sin(bx+c)dx = \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$

1081. $\int e^{ax}\cos bx\cos(bx+c)dx = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx+c) + 2b\cos(2bx+c)]}{2(a^2+4b^2)}$

1082. $\int e^{ax}\sin^n bx\,dx$

$$= \frac{1}{a^2+n^2b^2}\left[(a\sin bx - nb\cos bx)e^{ax}\sin^{n-1}bx + n(n-1)b^2\int e^{ax}\sin^{n-2}bx\,dx\right]$$

1083. $\int e^{ax}\cos^n bx\,dx$

$$= \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)e^{ax}\cos^{n-1}bx + n(n-1)b^2\int e^{ax}\cos^{n-2}bx\,dx\right]$$

1084. $\int xe^{ax}\sin bx\,dx$

$$= \frac{xe^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx - 2ab\cos bx]$$

1085. $\int xe^{ax}\cos bx\,dx$

$$= \frac{xe^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos bx + 2ab\sin bx]$$

1086. $\int x^m e^{ax}\sin bx\,dx$

$$= x^m e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2} - \frac{m}{a^2+b^2}\int x^{m-1}e^{ax}(a\sin bx - b\cos bx)\,dx$$

1087. $\int x^m e^{ax}\cos bx\,dx$

$$= x^m e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2} - \frac{m}{a^2+b^2}\int x^{m-1}e^{ax}(a\cos bx + b\sin bx)\,dx$$

1088. $\int \frac{e^{ax}}{\sin^n x}dx = -\frac{e^{ax}[a\sin x + (n-2)\cos x]}{(n-1)(n-2)\sin^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int \frac{e^{ax}}{\sin^{n-2}x}dx$

$(n \neq 1,2)$

1089. $\int \frac{e^{ax}}{\cos^n x}dx = -\frac{e^{ax}[a\cos x - (n-2)\sin x]}{(n-1)(n-2)\cos^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int \frac{e^{ax}}{\cos^{n-2}x}dx$

$(n \neq 1,2)$

1090. $\int e^{ax}\tan x\,dx = \frac{e^{ax}}{a}\tan x - \frac{1}{a}\int \frac{e^{ax}}{\cos^2 x}dx$

1091. $\int e^{ax}\tan^2 x\,dx = \frac{e^{ax}}{a}(a\tan x - 1) - a\int e^{ax}\tan x\,dx$

1092. $\int e^{ax}\tan^n x\,dx = \frac{e^{ax}}{n-1}\tan^{n-1}x - \frac{a}{n-1}\int e^{ax}\tan^{n-1}x\,dx - \int e^{ax}\tan^{n-2}x\,dx$

$(n \neq 1)$

1093. $\int e^{ax}\cot x\,dx = \frac{e^{ax}}{a}\cot x + \frac{1}{a}\int \frac{e^{ax}}{\sin^2 x}dx$

1094. $\int e^{ax}\cot^2 x\,dx = -\frac{e^{ax}}{a}(a\cot x + 1) + a\int e^{ax}\cot x\,dx$

1095. $\int e^{ax}\cot^n x\mathrm{d}x = -\frac{e^{ax}}{n-1}\cot^{n-1}x + \frac{a}{n-1}\int e^{ax}\cot^{n-1}x\mathrm{d}x - \int e^{ax}\cot^{n-2}x\mathrm{d}x$ $(n \neq 1)$

Ⅰ.1.4.3 双曲函数的积分

1096. $\int \sinh ax\mathrm{d}x = \frac{\cosh ax}{a}$

1097. $\int \sinh^2 ax\mathrm{d}x = \frac{\sinh 2ax}{4a} - \frac{x}{2}$

1098. $\int \sinh^n ax\mathrm{d}x = \frac{\sinh^{n-1}ax\cosh ax}{na} - \frac{n-1}{n}\int \sinh^{n-2}ax\mathrm{d}x$

1099. $\int \cosh ax\mathrm{d}x = \frac{\sinh ax}{a}$

1100. $\int \cosh^2 ax\mathrm{d}x = \frac{\sinh 2ax}{4a} + \frac{x}{2}$

1101. $\int \cosh^n ax\mathrm{d}x = \frac{\cosh^{n-1}ax\sinh ax}{na} + \frac{n-1}{n}\int \cosh^{n-2}ax\mathrm{d}x$

1102. $\int \tanh ax\mathrm{d}x = \frac{\ln(\cosh ax)}{a}$

1103. $\int \tanh^2 ax\mathrm{d}x = x - \frac{\tanh ax}{a}$

1104. $\int \tanh^n ax\mathrm{d}x = -\frac{\tanh^{n-1}ax}{(n-1)a} + \int(\tanh^{n-2}ax)\mathrm{d}x \quad (n \neq 1)$

1105. $\int \coth ax\mathrm{d}x = \frac{1}{a}\ln|\sinh ax|$

1106. $\int \coth^2 ax\mathrm{d}x = x - \frac{\coth ax}{a}$

1107. $\int \coth^n ax\mathrm{d}x = -\frac{\coth^{n-1}ax}{(n-1)a} + \int \coth^{n-2}ax\mathrm{d}x \quad (n \neq 1)$

1108. $\int \sinh^{2n}x\mathrm{d}x = (-1)^n\binom{2n}{n}\frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{\sinh(2n-2k)x}{2n-2k}$

1109. $\int \sinh^{2n+1}x\mathrm{d}x = \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\frac{\cosh(2n-2k+1)x}{2n-2k+1}$

1110. $\int \cosh^{2n}x\mathrm{d}x = \binom{2n}{n}\frac{x}{2^{2n}} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\frac{\sinh(2n-2k)x}{2n-2k}$

1111. $\int \cosh^{2n+1}x\mathrm{d}x = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}\frac{\sinh(2n-2k+1)x}{2n-2k+1}$

1112. $\int \tanh^{2n}x\mathrm{d}x = -\sum_{k=1}^{n}\frac{\tanh^{2n-2k+1}x}{2n-2k+1} + x$

1113. $\int \tanh^{2n+1} x\mathrm{d}x = -\sum_{k=1}^{n} \frac{\tanh^{2n-2k+2} x}{2n-2k+2} + \ln(\cosh x)$

1114. $\int \coth^{2n} x\mathrm{d}x = -\sum_{k=1}^{n} \frac{\coth^{2n-2k+1} x}{2n-2k+1} + x$

1115. $\int \coth^{2n+1} x\mathrm{d}x = -\sum_{k=1}^{n} \frac{\coth^{2n-2k+2} x}{2n-2k+2} + \ln|\sinh x|$

1116. $\int \operatorname{sech} x\mathrm{d}x = \arctan(\sinh x)$

1117. $\int \operatorname{sech}^2 x\mathrm{d}x = \tanh x$

1118. $\int \operatorname{csch} x\mathrm{d}x = \ln\left|\tanh \frac{x}{2}\right|$

1119. $\int \operatorname{csch}^2 x\mathrm{d}x = -\coth x$

1120. $\int \frac{\mathrm{d}x}{\sinh ax} = \frac{1}{a}\ln\left(\tanh \frac{ax}{2}\right)$

1121. $\int \frac{\mathrm{d}x}{\sinh^2 ax} = -\frac{1}{a}\coth ax$

1122. $\int \frac{\mathrm{d}x}{\sinh^n ax} = -\frac{\cosh ax}{(n-1)a\sinh^{n-1} ax} - \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\sinh^{n-2} ax} \quad (n \neq 1)$

1123. $\int \frac{\mathrm{d}x}{\cosh ax} = \frac{\arctan(\sinh ax)}{a}$

1124. $\int \frac{\mathrm{d}x}{\cosh^2 ax} = \frac{\tanh ax}{a}$

1125. $\int \frac{\mathrm{d}x}{\cosh^n ax} = \frac{\sinh ax}{(n-1)a\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\cosh^{n-2} ax} \quad (n \neq 1)$

1126. $$\int \frac{\mathrm{d}x}{\sinh^{2n} x} = \frac{\cosh x}{2n-1}\left[-\operatorname{csch}^{2n-1} x + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{2^k(n-1)(n-2)\cdots(n-k)}{(2n-3)(2n-5)\cdots(2n-2k-1)}\operatorname{csch}^{2n-2k-1} x\right]$$

1127. $$\int \frac{\mathrm{d}x}{\sinh^{2n+1} x} = \frac{\cosh x}{2n}\left[-\operatorname{csch}^{2n} x + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)}\operatorname{csch}^{2n-2k} x\right] + (-1)^n\frac{(2n-1)!!}{(2n)!!}\ln\left|\tanh \frac{x}{2}\right|$$

1128. $$\int \frac{\mathrm{d}x}{\cosh^{2n} x} = \frac{\sinh x}{2n-1}\left[\operatorname{sech}^{2n-1} x + \sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\cdots(n-k)}{(2n-3)(2n-5)\cdots(2n-2k-1)}\operatorname{sech}^{2n-2k-1} x\right]$$

1129. $\int \frac{\mathrm{d}x}{\cosh^{2n+1}x} = \frac{\sinh x}{2n}\Big[\operatorname{sech}^{2n}x + \sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)}\operatorname{sech}^{2n-2k}x\Big] + \frac{(2n-1)!!}{(2n)!!}\arctan(\sinh x)$

1130. $\int \frac{x}{\sinh^n ax}\mathrm{d}x = -\frac{x\cosh ax}{(n-1)a\sinh^{n-1}ax} - \frac{1}{(n-1)(n-2)a^2\sinh^{n-2}ax} - \frac{n-2}{n-1}\int\frac{x}{\sinh^{n-2}ax}\mathrm{d}x \quad (n\neq 1,2)$

1131. $\int \frac{x}{\cosh^n ax}\mathrm{d}x = \frac{x\sinh ax}{(n-1)a\cosh^{n-1}ax} + \frac{1}{(n-1)(n-2)a^2\cosh^{n-2}ax} + \frac{n-2}{n-1}\int\frac{x}{\cosh^{n-2}ax}\mathrm{d}x \quad (n\neq 1,2)$

1132. $\int \sinh ax\cosh ax\mathrm{d}x = \frac{\sinh^2 ax}{2a}$

1133. $\int \sinh^2 ax\cosh^2 ax\mathrm{d}x = \frac{\sinh 4ax}{32a} - \frac{x}{8}$

1134. $\int \sinh^n ax\cosh ax\mathrm{d}x = \frac{\sinh^{n+1}ax}{(n+1)a} \quad (n\neq -1)$

1135. $\int \sinh ax\cosh^n ax\mathrm{d}x = \frac{\cosh^{n+1}ax}{(n+1)a} \quad (n\neq -1)$

1136. $\int \sinh^p ax\cosh^q ax\mathrm{d}x = \frac{\sinh^{p+1}ax\cosh^{q-1}ax}{(p+q)a} + \frac{q-1}{p+q}\int\sinh^p ax\cosh^{q-2}ax\mathrm{d}x = \frac{\sinh^{p-1}ax\cosh^{q+1}ax}{(p+q)a} - \frac{p-1}{p+q}\int\sinh^{p-2}ax\cosh^q ax\mathrm{d}x$

1137. $\int \frac{\sinh^2 ax}{\cosh ax}\mathrm{d}x = \frac{1}{a}[\sinh ax - \arctan(\sinh ax)]$

1138. $\int \frac{\cosh^2 ax}{\sinh ax}\mathrm{d}x = \frac{1}{a}\left(\cosh ax + \ln\left|\tanh\frac{ax}{2}\right|\right)$

1139. $\int \frac{\sinh ax}{\cosh^n ax}\mathrm{d}x = -\frac{1}{(n-1)a\cosh ax} \quad (n\neq 1)$

1140. $\int \frac{\cosh ax}{\sinh^n ax}\mathrm{d}x = -\frac{1}{(n-1)a\sinh ax} \quad (n\neq 1)$

1141. $\int \frac{\sinh ax}{\cosh ax\pm 1}\mathrm{d}x = \frac{1}{a}\ln(\cosh ax\pm 1)$

1142. $\int \frac{\cosh ax}{1\pm\sinh ax}\mathrm{d}x = \pm\frac{1}{a}\ln|1\pm\sinh ax|$

1143. $\int \frac{\mathrm{d}x}{\sinh ax\cosh ax} = \frac{1}{a}\ln|\tanh ax|$

1144. $\int \frac{\mathrm{d}x}{\sinh^2 ax\cosh ax} = -\frac{1}{a}[\tan(\sinh ax) + \operatorname{csch} ax]$

1145. $\int \frac{\mathrm{d}x}{\sinh ax\cosh^2 ax} = \frac{1}{a}\left(\ln\left|\tanh\frac{ax}{2}\right| + \operatorname{sech} ax\right)$

1146. $\int \frac{\mathrm{d}x}{\sinh^2 ax\cosh^2 ax} = -\frac{2}{a}\coth 2ax$

1147. $\int \frac{\mathrm{d}x}{\sinh^p ax\cosh^q ax}$

$$= -\frac{1}{a(p-1)\sinh^{p-1} ax\cosh^{q-1} ax} - \frac{p+q-2}{p-1}\int \frac{\mathrm{d}x}{\sinh^{p-2} ax\cosh^q ax}$$

$$= \frac{1}{a(q-1)\sinh^{p-1} ax\cosh^{q-1} ax} + \frac{p+q-2}{q-1}\int \frac{\mathrm{d}x}{\sinh^p ax\cosh^{q-2} ax}$$

1148. $\int \sinh mx\sinh nx\mathrm{d}x = \frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)} \quad (m^2 \neq n^2)$

1149. $\int \sinh mx\cosh nx\mathrm{d}x = \frac{\cosh(m+n)x}{2(m+n)} + \frac{\cosh(m-n)x}{2(m-n)} \quad (m^2 \neq n^2)$

1150. $\int \cosh mx\cosh nx\mathrm{d}x = \frac{\sinh(m+n)x}{2(m+n)} + \frac{\sinh(m-n)x}{2(m-n)} \quad (m^2 \neq n^2)$

1151. $\int \sinh^2 x\cosh^2 x\mathrm{d}x = -\frac{x}{8} + \frac{1}{32}\sinh 4x$

1152. $\int \sinh^m x\cosh^n x\mathrm{d}x$

$$= \frac{\sinh^{m+1} x\cosh^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sinh^m x\cosh^{n-2} x\mathrm{d}x \quad (m+n \neq 0)$$

$$= \frac{\sinh^{m-1} x\cosh^{n+1} x}{m+n} - \frac{m-1}{m+n}\int \sinh^{m-2} x\cosh^n x\mathrm{d}x \quad (m+n \neq 0)$$

1153. $\int \operatorname{sech} x\tanh x\mathrm{d}x = -\operatorname{sech} x$

1154. $\int \operatorname{csch} x\coth x\mathrm{d}x = -\operatorname{csch} x$

1155. $\int \frac{\sinh x}{\cosh x}\mathrm{d}x = \int \tanh x\mathrm{d}x = \ln(\cosh x)$

1156. $\int \frac{\sinh^2 x}{\cosh^2 x}\mathrm{d}x = \int \tanh^2 x\mathrm{d}x = x - \tanh x$

1157. $\int \frac{\sinh^{2n+1}}{\cosh^m x}\mathrm{d}x$

$$= \sum_{\substack{k=0 \\ k\neq\frac{m-1}{2}}}^{n} (-1)^{n+k}\binom{n}{k}\frac{\cosh^{2k-m+1} x}{2k-m+1} + s(-1)^{n+\frac{m-1}{2}}\left(\frac{n}{\frac{m-1}{2}}\right)\ln(\cosh x)$$

（这里，m 为奇数并且 $m < 2n+1$ 时，$s = 1$；其他情况，$s = 0$）

1158. $\int \frac{\cosh x}{\sinh x}\mathrm{d}x = \int \coth x\mathrm{d}x = \ln|\sinh x|$

1159. $\int \frac{\cosh^2 x}{\sinh^2 x}\mathrm{d}x = \int \coth^2 x\mathrm{d}x = x - \coth x$

1160. $\int \frac{\cosh^{2n+1}x}{\sinh^m x}\mathrm{d}x = \sum\limits_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{n} \binom{n}{k} \frac{\sinh^{2k-m+1}x}{2k-m+1} + s\left(\frac{n}{\frac{m-1}{2}}\right)\ln|\sinh x|$

（这里，m 为奇数并且 $m<2n+1$ 时，$s=1$；其他情况，$s=0$）

1161. $\int \frac{\sinh 2x}{\sinh^2 x}\mathrm{d}x = 2\ln|\sinh x|$

1162. $\int \frac{\sinh 2x}{\sinh^n x}\mathrm{d}x = -\frac{2}{(n-2)\sinh^{n-2}x} \quad (n\neq 2)$

1163. $\int \frac{\sinh 2x}{\cosh^2 x}\mathrm{d}x = 2\ln(\cosh x)$

1164. $\int \frac{\sinh 2x}{\cosh^n x}\mathrm{d}x = -\frac{2}{(n-2)\cosh^{n-2}x} \quad (n\neq 2)$

1165. $\int \frac{\mathrm{d}x}{\sinh x\cosh x} = \ln|\tanh x|$

1166. $\int \frac{\mathrm{d}x}{\sinh^2 x\cosh x} = -\frac{1}{\sinh x} - \arctan(\sinh x)$

1167. $\int \frac{\mathrm{d}x}{\sinh x\cosh^2 x} = \frac{1}{\cosh x} + \ln\left|\tanh\frac{x}{2}\right|$

1168. $\int \frac{\mathrm{d}x}{\sinh^2 x\cosh^2 x} = -2\coth 2x$

1169. $\int \frac{\mathrm{d}x}{\sinh^{2m}x\cosh^{2n}x} = \sum\limits_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2n-2k-1}\binom{m+n-1}{k}\tanh^{2k-2m+1}x$

1170. $\int \frac{\mathrm{d}x}{\sinh^{2m+1}x\cosh^{2n+1}x}$

$$= \sum_{\substack{k=0\\ k\neq m}}^{m+n} \frac{(-1)^{k+1}}{2n-2k}\binom{m+n}{k}\tanh^{2k-2m}x + (-1)^m\binom{m+n}{m}\ln|\tanh x|$$

1171. $\int \sinh(ax+b)\sinh(cx+d)\mathrm{d}x$

$$= \frac{1}{2(a+c)}\sinh[(a+c)x+b+d] - \frac{1}{2(a-c)}\sinh[(a-c)x+b-d]$$

$(a^2\neq c^2)$

1172. $\int \sinh(ax+b)\cosh(cx+d)\mathrm{d}x$

$$= \frac{1}{2(a+c)}\cosh[(a+c)x+b+d] + \frac{1}{2(a-c)}\cosh[(a-c)x+b-d]$$

$(a^2\neq c^2)$

1173. $\int \cosh(ax+b)\cosh(cx+d)\mathrm{d}x$

$$= \frac{1}{2(a+c)}\sinh[(a+c)x+b+d] + \frac{1}{2(a-c)}\sinh[(a-c)x+b-d]$$

$(a^2 \neq c^2)$

1174. $\int \sinh(ax+b)\sinh(ax+d)\mathrm{d}x = -\frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)$

1175. $\int \sinh(ax+b)\cosh(ax+d)\mathrm{d}x = \frac{x}{2}\sinh(b-d) + \frac{1}{4a}\cosh(2ax+b+d)$

1176. $\int \cosh(ax+b)\cosh(ax+d)\mathrm{d}x = \frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)$

1177. $$\int \frac{\mathrm{d}x}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{a\tanh\frac{x}{2} - b + \sqrt{a^2+b^2}}{a\tanh\frac{x}{2} - b - \sqrt{a^2+b^2}}\right|$$

$$= \frac{2}{\sqrt{a^2+b^2}}\operatorname{artanh}\frac{a\tanh\frac{x}{2} - b}{\sqrt{a^2+b^2}}$$

1178. $$\int \frac{\mathrm{d}x}{a+b\cosh x} = \frac{1}{\sqrt{a^2-b^2}}\ln\left|\frac{a+b+\sqrt{a^2-b^2}\tanh\frac{x}{2}}{a+b-\sqrt{a^2-b^2}\tanh\frac{x}{2}}\right| \quad (a^2 > b^2)$$

$$= -\frac{2}{\sqrt{b^2-a^2}}\arcsin\frac{b+a\cosh x}{a+b\cosh x} \quad (a^2 < b^2)$$

1179. $$\int \frac{\mathrm{d}x}{a\cosh x + b\sinh x} = \begin{cases} \frac{1}{\sqrt{a^2-b^2}}\arctan\left|\sinh\left(x+\operatorname{artanh}\frac{b}{a}\right)\right| & (a > |b|) \\ \frac{1}{\sqrt{b^2-a^2}}\ln\left|\tanh\frac{x+\operatorname{artanh}\frac{a}{b}}{2}\right| & (b > |a|) \end{cases}$$

1180. $$\int \frac{\sinh x}{a\cosh x + b\sinh x}\mathrm{d}x = \begin{cases} \frac{a\ln\left[\cosh\left(x+\operatorname{artanh}\frac{b}{a}\right)\right] - bx}{a^2-b^2} & (a > |b|) \\ -\frac{a\ln\left|\sinh\left(x+\operatorname{artanh}\frac{a}{b}\right)\right| - bx}{b^2-a^2} & (b > |a|) \end{cases}$$

1181. $$\int \frac{\cosh x}{a\cosh x + b\sinh x}\mathrm{d}x = \begin{cases} -\frac{b\ln\left[\cosh\left(x+\operatorname{artanh}\frac{b}{a}\right)\right] - ax}{a^2-b^2} & (a > |b|) \\ \frac{b\ln\left|\sinh\left(x+\operatorname{artanh}\frac{a}{b}\right)\right| - ax}{b^2-a^2} & (b > |a|) \end{cases}$$

1182. $\int \frac{\mathrm{d}x}{\cosh x + \sinh x} = -\mathrm{e}^{-x} = \sinh x - \cosh x$

1183. $\int \frac{\mathrm{d}x}{\cosh x - \sinh x} = \mathrm{e}^{x} = \sinh x + \cosh x$

1184. $\int \frac{\sinh x}{\cosh x + \sinh x}\mathrm{d}x = \frac{x}{2} + \frac{1}{4}\mathrm{e}^{-2x}$

1185. $\int \frac{\sinh x}{\cosh x - \sinh x}\mathrm{d}x = -\frac{x}{2} + \frac{1}{4}\mathrm{e}^{2x}$

1186. $\int \frac{\cosh x}{\cosh x + \sinh x}\mathrm{d}x = \frac{x}{2} - \frac{1}{4}\mathrm{e}^{-2x}$

1187. $\int \frac{\cosh x}{\cosh x - \sinh x}\mathrm{d}x = \frac{x}{2} + \frac{1}{4}\mathrm{e}^{2x}$

1188. $\int \frac{\mathrm{d}x}{1 + \sinh^2 x} = \tanh x$

1189. $\int \frac{\mathrm{d}x}{1 - \sinh^2 x} = \begin{cases} \frac{1}{\sqrt{2}}\operatorname{artanh}(\sqrt{2}\tanh x) & (\sinh^2 x < 1) \\ \frac{1}{\sqrt{2}}\operatorname{arcoth}(\sqrt{2}\tanh x) & (\sinh^2 x > 1) \end{cases}$

1190. $\int \frac{\mathrm{d}x}{1 + \cosh^2 x} = \frac{1}{\sqrt{2}}\operatorname{arcoth}(\sqrt{2}\coth x)$

1191. $\int \frac{\mathrm{d}x}{1 - \cosh^2 x} = \coth x$

1192. $\int \sqrt{\tanh x}\,\mathrm{d}x = \operatorname{artanh}\sqrt{\tanh x} - \arctan\sqrt{\tanh x}$

1193. $\int \sqrt{\coth x}\,\mathrm{d}x = \operatorname{arcoth}\sqrt{\coth x} - \arctan\sqrt{\coth x}$

1194. $\int \frac{\sinh x}{\sqrt{a^2 + \sinh^2 x}}\mathrm{d}x$

$$= \begin{cases} \operatorname{arsinh}\frac{\cosh x}{\sqrt{a^2 - 1}} = \ln(\cosh x + \sqrt{a^2 + \sinh^2 x}) & (a^2 > 1) \\ \operatorname{arcosh}\frac{\cosh x}{\sqrt{1 - a^2}} = \ln(\cosh x + \sqrt{a^2 + \sinh^2 x}) & (a^2 < 1) \\ \ln(\cosh x) & (a^2 = 1) \end{cases}$$

1195. $\int \frac{\sinh x}{\sqrt{a^2 - \sinh^2 x}}\mathrm{d}x = \arcsin\frac{\cosh x}{\sqrt{a^2 + 1}} \quad (\sinh^2 x < a^2)$

1196. $\int \frac{\cosh x}{\sqrt{a^2 + \sinh^2 x}}\mathrm{d}x = \operatorname{arsinh}\frac{\sinh x}{a} = \ln(\sinh x + \sqrt{a^2 + \sinh^2 x})$

1197. $\int \frac{\cosh x}{\sqrt{a^2 - \sinh^2 x}}\mathrm{d}x = \arcsin\frac{\sinh x}{a} \quad (\sinh^2 x < a^2)$

1198. $\int \frac{\sinh x}{\sqrt{a^2 + \cosh^2 x}}\mathrm{d}x = \operatorname{arsinh}\frac{\cosh x}{a} = \ln(\cosh x + \sqrt{a^2 + \cosh^2 x})$

1199. $\int \frac{\sinh x}{\sqrt{a^2 - \cosh^2 x}}\mathrm{d}x = \arcsin\frac{\cosh x}{a} \quad (\cosh^2 x < a^2)$

1200. $\int \frac{\cosh x}{\sqrt{a^2 + \cosh^2 x}}\mathrm{d}x = \operatorname{arsinh}\frac{\sinh x}{\sqrt{a^2 + 1}} = \ln(\sinh x + \sqrt{a^2 + \cosh^2 x})$

1201. $\int \frac{\cosh x}{\sqrt{a^2-\cosh^2 x}}\mathrm{d}x = \arcsin\frac{\sinh}{\sqrt{a^2-1}} \quad (\cosh^2 x < a^2)$

1202. $\int \frac{\sinh x}{\sqrt{\sinh^2 x - a^2}}\mathrm{d}x = \operatorname{arcosh}\frac{\cosh x}{\sqrt{a^2+1}}$

$= \ln(\cosh x + \sqrt{\sinh^2 x - a^2}) \quad (\sinh^2 x > a^2)$

1203. $\int \frac{\sinh x}{\sqrt{\cosh^2 x - a^2}}\mathrm{d}x = \operatorname{arcosh}\frac{\cosh x}{a}$

$= \ln(\cosh x + \sqrt{\cosh^2 x - a^2}) \quad (\cosh^2 x > a^2)$

1204. $\int \frac{\cosh x}{\sqrt{\sinh^2 x - a^2}}\mathrm{d}x = \operatorname{arcosh}\frac{\sinh x}{a}$

$= \ln|\sinh x + \sqrt{\sinh^2 x - a^2}| \quad (\sinh^2 x > a^2)$

1205. $\int \frac{\cosh x}{\sqrt{\cosh^2 x - a^2}}\mathrm{d}x = \begin{cases} \operatorname{arcosh}\frac{\sinh x}{\sqrt{a^2-1}} & (a^2 > 1) \\ \ln|\sinh x| & (a^2 = 1) \end{cases}$

1206. $\int \frac{\mathrm{d}x}{\sqrt{\sinh 2ax}} = \frac{1}{2a}\mathrm{F}(r,\alpha) \quad (ax > 0)$

[这里，$\mathrm{F}(r,\alpha)$ 为第一类椭圆积分(见附录)，以下同. 式中，$r = \frac{1}{\sqrt{2}}$，$\alpha = \arccos\frac{1-\sinh 2ax}{1+\sinh 2ax}$]

1207. $\int \frac{\mathrm{d}x}{\sqrt{\cosh 2ax}} = \frac{1}{a\sqrt{2}}\mathrm{F}(r,\alpha) \quad (x \neq 0)$

$\left(\text{这里，} r = \frac{1}{\sqrt{2}},\ \alpha = \arcsin\sqrt{\frac{\cosh 2ax - 1}{\cosh 2ax}}\right)$

Ⅰ.1.4.4 双曲函数与幂函数、指数函数和三角函数组合的积分

1208. $\int x\sinh ax\mathrm{d}x = \frac{x\cosh ax}{a} - \frac{\sinh ax}{a^2}$

1209. $\int x^2\sinh ax\mathrm{d}x = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax$

1210. $\int x^m\sinh ax\mathrm{d}x = \frac{x^m\cosh ax}{a} - \frac{m}{a}\int x^{m-1}\cosh ax\mathrm{d}x$

1211. $\int x^m\sinh^n ax\mathrm{d}x = x^m\int\sinh^n ax\mathrm{d}x - mx^{m-1}\iint\sinh^n ax\mathrm{d}x\mathrm{d}x$

$+ m(m-1)x^{m-2}\iiint\sinh^n ax\mathrm{d}x\mathrm{d}x\mathrm{d}x - \cdots$

$$+(-1)^m m!\underbrace{\int\cdots\int}_{m+1个}\sinh^n ax\,dx\cdots dx$$

1212. $\int x\cosh ax\,dx=\frac{x\sinh ax}{a}-\frac{\cosh ax}{a^2}$

1213. $\int x^2\cosh ax\,dx=\left(\frac{x^2}{a}+\frac{2}{a^3}\right)\sinh ax-\frac{2x}{a^2}\cosh ax$

1214. $\int x^m\cosh ax\,dx=\frac{x^m\sinh ax}{a}-\frac{m}{a}\int x^{m-1}\sinh ax\,dx$

1215. $\int x^m\cosh^n ax\,dx=x^m\int\cosh^n ax\,dx-mx^{m-1}\iint\cosh^n ax\,dx\,dx$

$$+m(m-1)x^{m-2}\iiint\cosh^n ax\,dx\,dx\,dx-\cdots$$

$$+(-1)^m m!\underbrace{\int\cdots\int}_{m+1个}\cosh^n ax\,dx\cdots dx$$

1216. $\int\frac{\sinh ax}{x}dx=ax+\frac{(ax)^3}{3\cdot 3!}+\frac{(ax)^5}{5\cdot 5!}+\cdots$

1217. $\int\frac{\sinh ax}{x^2}dx=-\frac{\sinh ax}{x}+a\left[\ln|x|+\frac{(ax)^2}{2\cdot 2!}+\frac{(ax)^4}{4\cdot 4!}+\cdots\right]$

1218. $\int\frac{\sinh ax}{x^m}dx=-\frac{\sinh ax}{(m-1)x^{m-1}}+\frac{a}{m-1}\int\frac{\cosh ax}{x^{m-1}}dx\quad(m\neq 1)$

1219. $\int\frac{\cosh ax}{x}dx=\ln|x|+\frac{(ax)^2}{2\cdot 2!}+\frac{(ax)^4}{4\cdot 4!}+\cdots$

1220. $\int\frac{\cosh ax}{x^2}dx=-\frac{\cosh ax}{x}+a\left[ax+\frac{(ax)^3}{3\cdot 3!}+\frac{(ax)^5}{5\cdot 5!}+\cdots\right]$

1221. $\int\frac{\cosh ax}{x^m}dx=-\frac{\cosh ax}{(m-1)x^{m-1}}+\frac{a}{m-1}\int\frac{\sinh ax}{x^{m-1}}dx\quad(m\neq 1)$

1222. $\int x^{2m}\sinh x\,dx=(2m)!\left[\sum_{k=0}^{m}\frac{x^{2k}}{(2k)!}\cosh x-\sum_{k=0}^{m}\frac{x^{2k-1}}{(2k-1)!}\sinh x\right]$

1223. $\int x^{2m+1}\sinh x\,dx=(2m+1)!\sum_{k=0}^{m}\left[\frac{x^{2k+1}}{(2k+1)!}\cosh x-\frac{x^{2k}}{(2k)!}\sinh x\right]$

1224. $\int x^{2m}\cosh x\,dx=(2m)!\left[\sum_{k=0}^{m}\frac{x^{2k}}{(2k)!}\sinh x-\sum_{k=0}^{m}\frac{x^{2k-1}}{(2k-1)!}\cosh x\right]$

1225. $\int x^{2m+1}\cosh x\,dx=(2m+1)!\sum_{k=0}^{m}\left[\frac{x^{2k+1}}{(2k+1)!}\sinh x-\frac{x^{2k}}{(2k)!}\cosh x\right]$

1226. $\int\frac{\sinh x}{x^{2m}}dx=-\frac{1}{(2m-1)!x}\left[\sum_{k=0}^{m-2}\frac{(2k+1)!}{x^{2k+1}}\cosh x+\sum_{k=0}^{m-1}\frac{(2k)!}{x^{2k}}\sinh x\right]$

$$+\frac{1}{(2m-1)!}\mathrm{chi}(x)$$

[这里，chi(x)为双曲余弦积分(见附录)，以下同]

1227. $\int \frac{\sinh x}{x^{2m+1}}\mathrm{d}x$

$$= -\frac{1}{(2m)!x}\left[\sum_{k=0}^{m-1}\frac{(2k)!}{x^{2k}}\cosh x + \sum_{k=0}^{m-1}\frac{(2k+1)!}{x^{2k+1}}\sinh x\right] + \frac{1}{(2m)!}\mathrm{shi}(x)$$

[这里，shi(x)为双曲正弦积分(见附录)，以下同]

1228. $\int \frac{\cosh x}{x^{2m}}\mathrm{d}x = -\frac{1}{(2m-1)!x}\left[\sum_{k=0}^{m-2}\frac{(2k+1)!}{x^{2k+1}}\sinh x + \sum_{k=0}^{m-1}\frac{(2k)!}{x^{2k}}\cosh x\right]$

$$+ \frac{1}{(2m-1)!}\mathrm{shi}(x)$$

1229. $\int \frac{\cosh x}{x^{2m+1}}\mathrm{d}x$

$$= -\frac{1}{(2m)!x}\left[\sum_{k=0}^{m-1}\frac{(2k)!}{x^{2k}}\sinh x + \sum_{k=0}^{m-1}\frac{(2k+1)!}{x^{2k+1}}\cosh x\right] + \frac{1}{(2m)!}\mathrm{chi}(x)$$

1230. $\int \frac{x^m}{\sinh x}\mathrm{d}x = \sum_{k=0}^{\infty}\frac{(2-2^{2k})B_{2k}}{(m+2k)(2k)!}x^{m+2k} \quad (|x|<\pi,\ m>0)$

[这里，B_{2k} 为伯努利数(见附录)，以下同]

1231. $\int \frac{x^m}{\cosh x}\mathrm{d}x = \sum_{k=0}^{\infty}\frac{E_{2k}}{(m+2k+1)(2k)!}x^{m+2k+1} \quad \left(|x|<\frac{\pi}{2},\ m\geqslant 0\right)$

[这里，E_{2k} 为欧拉数(见附录)]

1232. $\int x^p \tanh x\mathrm{d}x = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(p+2k)(2k)!}x^{p+2k} \quad \left(p\geqslant -1,\ |x|<\frac{\pi}{2}\right)$

1233. $\int x^p \coth x\mathrm{d}x = \sum_{k=0}^{\infty}\frac{2^{2k}B_{2k}}{(p+2k)(2k)!}x^{p+2k} \quad (p\geqslant 1,\ |x|<\pi)$

1234. $\int (a+bx)\sinh kx\mathrm{d}x = \frac{1}{k}(a+bx)\cosh kx - \frac{b}{k^2}\sinh kx$

1235. $\int (a+bx)\cosh kx\mathrm{d}x = \frac{1}{k}(a+bx)\sinh kx - \frac{b}{k^2}\cosh kx$

1236. $\int (a+bx)^2\sinh kx\mathrm{d}x = \frac{1}{k}\left[(a+bx)^2 + \frac{2b^2}{k^2}\right]\cosh kx - \frac{2b(a+bx)}{k^2}\sinh kx$

1237. $\int (a+bx)^2\cosh kx\mathrm{d}x = \frac{1}{k}\left[(a+bx)^2 + \frac{2b^2}{k^2}\right]\sinh kx - \frac{2b(a+bx)}{k^2}\cosh kx$

1238. $\int \mathrm{e}^{ax}\sinh(ax+c)\mathrm{d}x = -\frac{1}{2}x\mathrm{e}^{-c} + \frac{1}{4a}\mathrm{e}^{2ax+c}$

1239. $\int \mathrm{e}^{ax}\cosh(ax+c)\mathrm{d}x = \frac{1}{2}x\mathrm{e}^{-c} + \frac{1}{4a}\mathrm{e}^{2ax+c}$

1240. $\int x\mathrm{e}^{ax}\sinh ax\mathrm{d}x = \frac{\mathrm{e}^{2ax}}{4a}\left(x-\frac{1}{2a}\right) - \frac{x^2}{4}$

1241. $\int x\mathrm{e}^{ax}\cosh ax\mathrm{d}x = \frac{\mathrm{e}^{2ax}}{4a}\left(x-\frac{1}{2a}\right) + \frac{x^2}{4}$

1242. $\int x^2 e^{ax}\sinh ax\,dx = \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right) - \frac{x^3}{6}$

1243. $\int x^2 e^{ax}\cosh ax\,dx = \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right) + \frac{x^3}{6}$

1244. $\int e^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2 - b^2}(a\sinh bx - b\cosh bx) \quad (a^2 \neq b^2)$

1245. $\int e^{ax}\cosh bx\,dx = \frac{e^{ax}}{a^2 - b^2}(a\cosh bx - b\sinh bx) \quad (a^2 \neq b^2)$

1246. $\int x e^{ax}\sinh bx\,dx$

$$= \frac{e^{ax}}{a^2 - b^2}\left[\left(ax - \frac{a^2 + b^2}{a^2 - b^2}\right)\sinh bx - \left(bx - \frac{2ab}{a^2 - b^2}\right)\cosh bx\right] \quad (a^2 \neq b^2)$$

1247. $\int x e^{ax}\cosh bx\,dx$

$$= \frac{e^{ax}}{a^2 - b^2}\left[\left(ax - \frac{a^2 + b^2}{a^2 - b^2}\right)\cosh bx - \left(bx - \frac{2ab}{a^2 - b^2}\right)\sinh bx\right] \quad (a^2 \neq b^2)$$

1248. $\int x^2 e^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2 - b^2}\left\{\left[ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2}x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2}\right]\sinh bx\right.$

$$\left. - \left[bx^2 - \frac{4ab}{a^2 - b^2}x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2}\right]\cosh x\right\} \quad (a^2 \neq b^2)$$

1249. $\int x^2 e^{ax}\cosh bx\,dx = \frac{e^{ax}}{a^2 - b^2}\left\{\left[ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2}x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2}\right]\cosh bx\right.$

$$\left. - \left[bx^2 - \frac{4ab}{a^2 - b^2}x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2}\right]\sinh x\right\} \quad (a^2 \neq b^2)$$

1250. $\int \frac{e^{ax}\sinh bx}{x}dx = \frac{1}{2}\{\mathrm{Ei}[(a + b)x] - \mathrm{Ei}[(a - b)x]\} \quad (a^2 \neq b^2)$

[这里，Ei(ax)为指数积分(见附录)，以下同]

1251. $\int \frac{e^{ax}\cosh bx}{x}dx = \frac{1}{2}\{\mathrm{Ei}[(a + b)x] + \mathrm{Ei}[(a - b)x]\} \quad (a^2 \neq b^2)$

1252. $\int \frac{e^{ax}\sinh bx}{x^2}dx$

$$= \frac{1}{2}\{(a + b)\mathrm{Ei}[(a + b)x] - (a - b)\mathrm{Ei}[(a - b)x]\} - \frac{e^{ax}\sinh bx}{2x}$$

$(a^2 \neq b^2)$

1253. $\int \frac{e^{ax}\cosh bx}{x^2}dx$

$$= \frac{1}{2}\{(a + b)\mathrm{Ei}[(a + b)x] + (a - b)\mathrm{Ei}[(a - b)x]\} - \frac{e^{ax}\cosh bx}{2x}$$

$(a^2 \neq b^2)$

1254. $\int \sinh ax\sin bx\,dx = \frac{1}{a^2 + b^2}(a\cosh ax\sin bx - b\sinh ax\cos bx)$

1255. $\int \sinh ax\cos bx\,\mathrm{d}x = \frac{1}{a^2+b^2}(a\cosh ax\cos bx + b\sinh ax\sin bx)$

1256. $\int \cosh ax\sin bx\,\mathrm{d}x = \frac{1}{a^2+b^2}(a\sinh ax\sin bx - b\cosh ax\cos bx)$

1257. $\int \cosh ax\cos bx\,\mathrm{d}x = \frac{1}{a^2+b^2}(a\sinh ax\cos bx + b\cosh ax\sin bx)$

Ⅰ.1.4.5 反双曲函数的积分

1258. $\int \operatorname{arsinh}\frac{x}{a}\mathrm{d}x = x\operatorname{arsinh}\frac{x}{a} - \sqrt{x^2+a^2} \quad (a>0)$

1259. $\int x\operatorname{arsinh}\frac{x}{a}\mathrm{d}x = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\operatorname{arsinh}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2} \quad (a>0)$

1260. $\int x^2\operatorname{arsinh}\frac{x}{a}\mathrm{d}x = \frac{x^3}{3}\operatorname{arsinh}\frac{x}{a} - \frac{2a^2-x^2}{9}\sqrt{x^2+a^2}$

1261. $\int x^m\operatorname{arsinh}\frac{x}{a}\mathrm{d}x = \frac{x^{m+1}}{m+1}\operatorname{arsinh}\frac{x}{a} - \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2+a^2}}\mathrm{d}x \quad (m\neq -1)$

1262. $\int \frac{\operatorname{arsinh}\frac{x}{a}}{x}\mathrm{d}x = \frac{x}{a} - \frac{1}{2\cdot3\cdot3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot3}{2\cdot4\cdot5\cdot5}\left(\frac{x}{a}\right)^5$
$- \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot7}\left(\frac{x}{a}\right)^7 + \cdots \quad (x^2<a^2)$

1263. $\int \frac{\operatorname{arsinh}\frac{x}{a}}{x^2}\mathrm{d}x = -\frac{1}{x}\operatorname{arsinh}\frac{x}{a} - \frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$

1264. $\int \frac{\operatorname{arsinh}\frac{x}{a}}{x^m}\mathrm{d}x = -\frac{1}{(m-1)x^{m-1}}\operatorname{arsinh}\frac{x}{a} + \frac{1}{m-1}\int\frac{\mathrm{d}x}{x^{m-1}\sqrt{a^2+x^2}} \quad (m\neq 1)$

1265. $\int \operatorname{arcosh}\frac{x}{a}\mathrm{d}x = x\operatorname{arcosh}\frac{x}{a} \mp \sqrt{x^2-a^2}$

$\left(\text{当 } \operatorname{arcosh}\frac{x}{a}>0 \text{ 时,取 } - \text{ 号;当 } \operatorname{arcosh}\frac{x}{a}<0 \text{ 时,取 } + \text{ 号}\right)$

1266. $\int x\operatorname{arcosh}\frac{x}{a}\mathrm{d}x = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{arcosh}\frac{x}{a} \mp \frac{x}{4}\sqrt{x^2-a^2}$

$\left(\text{当 } \operatorname{arcosh}\frac{x}{a}>0 \text{ 时,取 } - \text{ 号;当 } \operatorname{arcosh}\frac{x}{a}<0 \text{ 时,取 } + \text{ 号}\right)$

1267. $\int x^2\operatorname{arcosh}\frac{x}{a}\mathrm{d}x = \frac{x^3}{3}\operatorname{arcosh}\frac{x}{a} \mp \frac{2a^2+x^2}{9}\sqrt{x^2-a^2}$

$\left(\text{当 } \operatorname{arcosh}\frac{x}{a}>0 \text{ 时,取 } - \text{ 号;当 } \operatorname{arcosh}\frac{x}{a}<0 \text{ 时,取 } + \text{ 号}\right)$

1268. $\int x^m \operatorname{arcosh}\frac{x}{a}\mathrm{d}x = \frac{x^{m+1}}{m+1}\operatorname{arcosh}\frac{x}{a} \mp \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2-a^2}}\mathrm{d}x \quad (m \neq -1)$

$\left(当 \operatorname{arcosh}\frac{x}{a} > 0 时,取 - 号;当 \operatorname{arcosh}\frac{x}{a} < 0 时,取 + 号\right)$

1269. $\int\frac{\operatorname{arcosh}\frac{x}{a}}{x}\mathrm{d}x = \mp\left[\frac{1}{2}\left(\ln\left|\frac{2x}{a}\right|\right)^2 + \frac{1}{2\cdot2\cdot2}\left(\frac{x}{a}\right)^2 + \frac{1\cdot3}{2\cdot4\cdot4\cdot4}\left(\frac{x}{a}\right)^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot6\cdot6}\left(\frac{x}{a}\right)^6 + \cdots\right]$

$\left(当 \operatorname{arcosh}\frac{x}{a} < 0 时,取 - 号;当 \operatorname{arcosh}\frac{x}{a} > 0 时,取 + 号\right)$

1270. $\int\frac{\operatorname{arcosh}\frac{x}{a}}{x^2}\mathrm{d}x = -\frac{1}{x}\operatorname{arcosh}\frac{x}{a} \mp \frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$

$\left(当 \operatorname{arcosh}\frac{x}{a} < 0 时,取 - 号;当 \operatorname{arcosh}\frac{x}{a} > 0 时,取 + 号\right)$

1271. $\int\frac{\operatorname{arcosh}\frac{x}{a}}{x^m}\mathrm{d}x = -\frac{1}{(m-1)x^{m-1}}\operatorname{arcosh}\frac{x}{a} + \frac{1}{m-1}\int\frac{\mathrm{d}x}{x^{m-1}\sqrt{x^2-a^2}}$

$(m \neq 1)$

1272. $\int\operatorname{artanh}\frac{x}{a}\mathrm{d}x = x\operatorname{artanh}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2) \quad \left(\left|\frac{x}{a}\right| < 1\right)$

1273. $\int x\operatorname{artanh}\frac{x}{a}\mathrm{d}x = \frac{x^2-a^2}{2}\operatorname{artanh}\frac{x}{a} + \frac{ax}{2} \quad \left(\left|\frac{x}{a}\right| < 1\right)$

1274. $\int x^m\operatorname{artanh}\frac{x}{a}\mathrm{d}x = \frac{x^{m+1}}{m+1}\operatorname{artanh}\frac{x}{a} - \frac{a}{m+1}\int\frac{x^{m+1}}{a^2-x^2}\mathrm{d}x \quad (m \neq -1)$

1275. $\int\frac{\operatorname{artanh}\frac{x}{a}}{x}\mathrm{d}x = \frac{1}{1^2}\frac{x}{a} + \frac{1}{3^2}\left(\frac{x}{a}\right)^3 + \frac{1}{5^2}\left(\frac{x}{a}\right)^5 + \frac{1}{7^2}\left(\frac{x}{a}\right)^7 + \cdots$

1276. $\int\frac{\operatorname{artanh}\frac{x}{a}}{x^2}\mathrm{d}x = -\frac{1}{a}\left(\frac{a}{x}\operatorname{artanh}\frac{x}{a} + \ln\left|\frac{\sqrt{a^2-x^2}}{x}\right|\right)$

1277. $\int\frac{\operatorname{artanh}\frac{x}{a}}{x^3}\mathrm{d}x = -\frac{1}{2x^2}\left(\frac{x}{a} - \frac{x^2-a^2}{a^2}\operatorname{artanh}\frac{x}{a}\right)$

1278. $\int\frac{\operatorname{artanh}\frac{x}{a}}{x^m}\mathrm{d}x = -\frac{1}{(m-1)x^{m-1}}\operatorname{artanh}\frac{x}{a} + \frac{a}{m-1}\int\frac{\mathrm{d}x}{(a^2-x^2)x^{m-1}}$

1279. $\int\operatorname{arcoth}\frac{x}{a}\mathrm{d}x = x\operatorname{arcoth}\frac{x}{a} + \frac{a}{2}\ln(x^2-a^2) \quad \left(\left|\frac{x}{a}\right| > 1\right)$

1280. $\int x\operatorname{arcoth}\frac{x}{a}\mathrm{d}x = \frac{x^2-a^2}{2}\operatorname{arcoth}\frac{x}{a} + \frac{ax}{2} \quad \left(\left|\frac{x}{a}\right| > 1\right)$

1281. $\int x^m \operatorname{arcoth}\frac{x}{a}\mathrm{d}x = \frac{x^{m+1}}{m+1}\operatorname{arcoth}\frac{x}{a} + \frac{a}{m+1}\int\frac{x^{m+1}}{x^2-a^2}\mathrm{d}x \quad (m \neq -1)$

1282. $\int\frac{\operatorname{arcoth}\frac{x}{a}}{x}\mathrm{d}x = -\frac{1}{1^2}\left(\frac{x}{a}\right)^1 - \frac{1}{3^2}\left(\frac{x}{a}\right)^3 - \frac{1}{5^2}\left(\frac{x}{a}\right)^5 - \frac{1}{7^2}\left(\frac{x}{a}\right)^7 - \cdots$

1283. $\int\frac{\operatorname{arcoth}\frac{x}{a}}{x^2}\mathrm{d}x = -\frac{1}{a}\left(\frac{a}{x}\operatorname{arcoth}\frac{x}{a} + \ln\frac{\sqrt{a^2-x^2}}{x}\right)$

1284. $\int\frac{\operatorname{arcoth}\frac{x}{a}}{x^3}\mathrm{d}x = -\frac{1}{2x^2}\left(\frac{x}{a} - \frac{x^2-a^2}{a^2}\operatorname{arcoth}\frac{x}{a}\right)$

1285. $\int\frac{\operatorname{arcoth}\frac{x}{a}}{x^m}\mathrm{d}x = -\frac{1}{(m-1)x^{m-1}}\operatorname{arcoth}\frac{x}{a} + \frac{a}{m-1}\int\frac{\mathrm{d}x}{(a^2-x^2)x^{m-1}}$

1286. $\int\operatorname{arsech}x\,\mathrm{d}x = x\operatorname{arsech}x + \arcsin x$

1287. $\int x\operatorname{arsech}x\,\mathrm{d}x = \frac{x^2}{2}\operatorname{arsech}x - \frac{1}{2}\sqrt{1-x^2}$

1288. $\int x^n\operatorname{arsech}x\,\mathrm{d}x = \frac{x^{n+1}}{n+1}\operatorname{arsech}x + \frac{1}{n+1}\int\frac{x^n}{\sqrt{1-x^2}}\mathrm{d}x \quad (n \neq -1)$

1289. $\int\operatorname{arcsch}x\,\mathrm{d}x = x\operatorname{arcsch}x + \frac{x}{|x|}\operatorname{arsinh}x$

1290. $\int x\operatorname{arcsch}x\,\mathrm{d}x = \frac{x^2}{2}\operatorname{arcsch}x + \frac{1}{2}\frac{x}{|x|}\sqrt{1+x^2}$

1291. $\int x^n\operatorname{arcsch}x\,\mathrm{d}x = \frac{x^{n+1}}{n+1}\operatorname{arcsch}x + \frac{1}{n+1}\frac{x}{|x|}\int\frac{x^n}{\sqrt{1+x^2}}\mathrm{d}x \quad (n \neq -1)$

Ⅰ.2 特殊函数的不定积分

Ⅰ.2.1 完全椭圆积分的积分

设 $k' = \sqrt{1-k^2}$，并且 $k^2 < 1$.

1. $\int \mathrm{K}(k)\mathrm{d}k = \frac{k\pi}{2}\left\{1 + \sum_{j=1}^{\infty}\frac{[(2j)!]^2k^{2j}}{(2j+1)2^{4j}(j!)^4}\right\}$

[这里，$\mathrm{K}(k) = \mathrm{F}\left(k, \frac{\pi}{2}\right)$ 为第一类完全椭圆积分(见附录)，以下同]

2. $\int E(k)dk = \frac{k\pi}{2}\left\{1 - \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(4j^2-1)2^{4j}(j!)^4}\right\}$

[这里，$E(k) = E\left(k, \frac{\pi}{2}\right)$ 为第二类完全椭圆积分(见附录)，以下同]

3. $\int K(k)k\,dk = E(k) - (1-k^2)K(k)$

4. $\int E(k)k\,dk = \frac{1}{3}[(1+k^2)E(k) - (1-k^2)K(k)]$

5. $\int \frac{K(k)}{k^2}dk = -\frac{E(k)}{k}$

6. $\int \frac{E(k)}{k^2}dk = \frac{1}{k}[(1-k^2)K(k) - 2E(k)]$

7. $\int \frac{E(k)}{1-k^2}dk = kK(k)$

8. $\int \frac{kE(k)}{1-k^2}dk = K(k) - E(k)$

9. $\int \frac{K(k)-E(k)}{k}dk = -E(k)$

10. $\int \frac{E(k)-(1-k^2)K(k)}{k}dk = 2E(k) - (1-k^2)K(k)$

11. $\int \frac{(1+k^2)K(k)-E(k)}{k}dk = -(1-k^2)K(k)$

Ⅰ.2.2 勒让德椭圆积分(不完全椭圆积分)的积分

12. $\int_0^x \frac{F(k,x)}{\sqrt{1-k^2\sin^2 x}}dx = \frac{1}{2}[F(k,x)]^2 \quad \left(0 < x \leqslant \frac{\pi}{2}\right)$

[这里，$F(k,x)$ 为第一类勒让德椭圆积分(见附录)，以下同]

13. $\int_0^x E(k,x)\sqrt{1-k^2\sin^2 x}\,dx = \frac{1}{2}[E(k,x)]^2$

[这里，$E(k,x)$ 为第二类勒让德椭圆积分(见附录)，以下同]

14. $\int_0^x F(k,x)\sin x\,dx = -\cos x F(k,x) + \frac{1}{k}\arcsin(k\sin x)$

15. $\int_0^x F(k,x)\cos x\,dx$

$$= \sin x F(k,x) + \frac{1}{k}\operatorname{arcosh}\sqrt{\frac{1-k^2\sin^2 x}{1-k^2}} - \frac{1}{k}\operatorname{arcosh}\frac{1}{\sqrt{1-k^2}}$$

16. $\int_0^x \mathrm{E}(k,x)\sin x\mathrm{d}x$

$$= -\cos x\mathrm{E}(k,x) + \frac{1}{2k}\left[k\sin x\sqrt{1-k^2\sin^2 x} + \arcsin(k\sin x)\right]$$

17. $\int_0^x \mathrm{E}(k,x)\cos x\mathrm{d}x = \sin x\mathrm{E}(k,x) + \frac{1}{2k}\Big[k\cos x\sqrt{1-k^2\sin^2 x}$

$$-(1-k^2)\operatorname{arcosh}\sqrt{\frac{1-k^2\sin^2 x}{1-k^2}} - k$$

$$+(1-k^2)\operatorname{arcosh}\frac{1}{\sqrt{1-k^2}}\Big]$$

18. $\int \mathrm{F}(k,x)k\mathrm{d}k = \mathrm{E}(k,x) - (1-k^2)\mathrm{F}(k,x) + (\sqrt{1-k^2\sin^2 x} - 1)\cot x$

19. $\int \mathrm{E}(k,x)k\mathrm{d}k$

$$= \frac{1}{3}\left[(1+k^2)\mathrm{E}(k,x) - (1-k^2)\mathrm{F}(k,x) + (\sqrt{1-k^2\sin^2 x} - 1)\cot x\right]$$

Ⅰ.2.3 指数积分函数的积分

20. $\int_x^\infty \mathrm{Ei}(-\alpha x)\mathrm{Ei}(-\beta x)\mathrm{d}x = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\mathrm{Ei}[-(\alpha+\beta)x] - x\mathrm{Ei}(-\alpha x)\mathrm{Ei}(-\beta x)$

$$-\frac{\mathrm{e}^{-\alpha x}}{\alpha}\mathrm{Ei}(-\beta x) - \frac{\mathrm{e}^{-\beta x}}{\beta}\mathrm{Ei}(-\alpha x) \quad [\mathrm{Re}(\alpha+\beta)>0]$$

[这里，Ei(z)为指数积分(见附录)，以下同]

21. $\int_x^\infty \frac{\mathrm{Ei}[-a(x+b)]}{x^2}\mathrm{d}x = \left(\frac{1}{x}+\frac{1}{b}\right)\mathrm{Ei}[-a(x+b)] - \frac{\mathrm{e}^{-ab}}{b}\mathrm{Ei}(-ax)$

$(a>0,\ b>0)$

22. $\int_x^\infty \frac{\mathrm{Ei}[-a(x+b)]}{x^{n+1}}\mathrm{d}x$

$$= \left[\frac{1}{x^n} - \frac{(-1)^n}{b^n}\right]\frac{\mathrm{Ei}[-a(x+b)]}{n} + \frac{\mathrm{e}^{-ab}}{n}\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{b^{n-k}}\int_x^\infty \frac{\mathrm{e}^{-ax}}{x^{k+1}}\mathrm{d}x$$

$(a>0,\ b>0)$

23. $\int_0^x \mathrm{e}^x\mathrm{Ei}(-x)\mathrm{d}x = \mathrm{e}^x\mathrm{Ei}(-x) - \ln x - \gamma$

[这里，γ 是欧拉常数(见附录)]

24. $\int_0^x \mathrm{e}^{-\beta x}\mathrm{Ei}(-\alpha x)\mathrm{d}x = -\frac{1}{\beta}\left\{\mathrm{e}^{-\beta x}\mathrm{Ei}(-\alpha x) + \ln\left(1+\frac{\beta}{\alpha}\right) - \mathrm{Ei}[-(\alpha+\beta)x]\right\}$

Ⅰ.2.4 正弦积分和余弦积分函数的积分

25. $\int \sin\alpha x \operatorname{si}(\beta x)\mathrm{d}x = -\dfrac{\cos\alpha x \operatorname{si}(\beta x)}{\alpha} + \dfrac{\operatorname{si}(\alpha x+\beta x)-\operatorname{si}(\alpha x-\beta x)}{2\alpha}$

[这里，si(z)为正弦积分(见附录)，以下同]

26. $\int \cos\alpha x \operatorname{si}(\beta x)\mathrm{d}x = \dfrac{\sin\alpha x \operatorname{si}(\beta x)}{\alpha} + \dfrac{\operatorname{ci}(\alpha x+\beta x)-\operatorname{ci}(\alpha x-\beta x)}{2\alpha}$

[这里，ci(z)为余弦积分(见附录)，以下同]

27. $\int \sin\alpha x \operatorname{ci}(\beta x)\mathrm{d}x = -\dfrac{\cos\alpha x \operatorname{ci}(\beta x)}{\alpha} + \dfrac{\operatorname{ci}(\alpha x+\beta x)+\operatorname{ci}(\alpha x-\beta x)}{2\alpha}$

28. $\int \cos\alpha x \operatorname{ci}(\beta x)\mathrm{d}x = \dfrac{\sin\alpha x \operatorname{ci}(\beta x)}{\alpha} - \dfrac{\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)}{2\alpha}$

29. $$\begin{aligned}\int \operatorname{si}(\alpha x)\operatorname{si}(\beta x)\mathrm{d}x = {} & x\operatorname{si}(\alpha x)\operatorname{si}(\beta x) - \frac{1}{2\beta}[\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)] \\ & - \frac{1}{2\alpha}[\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x-\alpha x)] \\ & + \frac{1}{\alpha}\cos\alpha x \operatorname{si}(\beta x) + \frac{1}{\beta}\cos\beta x \operatorname{si}(\alpha x)\end{aligned}$$

30. $$\begin{aligned}\int \operatorname{si}(\alpha x)\operatorname{ci}(\beta x)\mathrm{d}x = {} & x\operatorname{si}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{\alpha}\cos\alpha x \operatorname{ci}(\beta x) - \frac{1}{\beta}\sin\beta x \operatorname{si}(\alpha x) \\ & - \left(\frac{1}{2\alpha}+\frac{1}{2\beta}\right)\operatorname{ci}(\alpha x+\beta x) - \left(\frac{1}{2\alpha}-\frac{1}{2\beta}\right)\operatorname{ci}(\alpha x-\beta x)\end{aligned}$$

31. $$\begin{aligned}\int \operatorname{ci}(\alpha x)\operatorname{ci}(\beta x)\mathrm{d}x = {} & x\operatorname{ci}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{2\alpha}[\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)] \\ & + \frac{1}{2\beta}[\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x-\alpha x)] \\ & - \frac{1}{\alpha}\sin\alpha x \operatorname{ci}(\beta x) - \frac{1}{\beta}\sin\beta x \operatorname{ci}(\alpha x)\end{aligned}$$

32. $\int_x^\infty \dfrac{\operatorname{si}[a(x+b)]}{x^2}\mathrm{d}x = \left(\dfrac{1}{x}+\dfrac{1}{b}\right)\operatorname{si}[a(x+b)] - \dfrac{\cos ab \operatorname{si}(ax)+\sin ab \operatorname{ci}(ax)}{b}$

$(a>0,\ b>0)$

33. $\int_x^\infty \dfrac{\operatorname{ci}[a(x+b)]}{x^2}\mathrm{d}x = \left(\dfrac{1}{x}+\dfrac{1}{b}\right)\operatorname{ci}[a(x+b)] + \dfrac{\sin ab \operatorname{si}(ax)-\cos ab \operatorname{ci}(ax)}{b}$

$(a>0,\ b>0)$

Ⅰ.2.5 概率积分和菲涅耳函数的积分

34. $\int \Phi(ax)\mathrm{d}x = x\,\Phi(ax) + \dfrac{\mathrm{e}^{-a^2x^2}}{a\sqrt{\pi}}$

[这里，$\Phi(x)$为概率积分(见附录)]

35. $\int \mathrm{S}(ax)\mathrm{d}x = x\,\mathrm{S}(ax) + \dfrac{\cos a^2x^2}{a\sqrt{2\pi}}$

[这里，$\mathrm{S}(x)$为菲涅耳函数(见附录)]

36. $\int \mathrm{C}(ax)\mathrm{d}x = x\,\mathrm{C}(ax) - \dfrac{\sin a^2x^2}{a\sqrt{2\pi}}$

[这里，$\mathrm{C}(x)$为菲涅耳函数(见附录)]

Ⅰ.2.6 贝塞尔函数的积分

37. $\int \mathrm{J}_p(x)\mathrm{d}x = 2\sum_{k=0}^{\infty}\mathrm{J}_{p+2k+1}(x)$

38. $\int x^{p+1}\mathrm{Z}_p(x)\mathrm{d}x = x^{p+1}\mathrm{Z}_{p+1}(x)$

39. $\int x^{-p+1}\mathrm{Z}_p(x)\mathrm{d}x = -x^{-p+1}\mathrm{Z}_{p-1}(x)$

40. $\int x[\mathrm{Z}_p(ax)]^2\mathrm{d}x = \dfrac{x^2}{2}\{[\mathrm{Z}_p(ax)]^2 - \mathrm{Z}_{p-1}(ax)\mathrm{Z}_{p+1}(ax)\}$

41. $\int x\mathrm{Z}_p(ax)\mathrm{W}_p(bx)\mathrm{d}x = \dfrac{bx\mathrm{Z}_p(ax)\mathrm{W}_{p-1}(bx) - ax\mathrm{Z}_{p-1}(ax)\mathrm{W}_p(bx)}{a^2 - b^2}$

42. $\int \dfrac{\mathrm{Z}_p(ax)\mathrm{W}_q(ax)}{x}\mathrm{d}x$

$$= \frac{ax[\mathrm{Z}_{p-1}(ax)\mathrm{W}_q(ax) - \mathrm{Z}_p(ax)\mathrm{W}_{q-1}(ax)]}{p^2 - q^2} - \frac{\mathrm{Z}_p(ax)\mathrm{W}_q(ax)}{p+q}$$

43. $\int \mathrm{Z}_1(x)\mathrm{d}x = -\mathrm{Z}_0(x)$

44. $\int x\mathrm{Z}_0(x)\mathrm{d}x = x\mathrm{Z}_1(x)$

[上述诸式中，$\mathrm{Z}_p(x)$，$\mathrm{W}_p(x)$均为任意贝塞尔函数]

Ⅱ 定积分表

在定积分中，若被积函数在积分区间内有奇点，则其积分值被视为积分主值.

Ⅱ.1 初等函数的定积分

Ⅱ.1.1 幂函数和代数函数的定积分

与不定积分一样，公式中出现的变量与常量，都应在使公式两边都有意义的范围之内. 当没有特别说明时，l,m,n 为非零的正整数；$a,b,c,d,p,q,\alpha,\beta,\gamma$ 是非零的实数.

Ⅱ.1.1.1 含有 x^n 和 $a^p \pm x^p$ 的积分

1. $\int_1^{\infty} \frac{\mathrm{d}x}{x^m} = \frac{1}{m-1} \quad (m > 1)$

2. $\int_0^a x^m (a-x)^n \mathrm{d}x = \frac{m!\,n!\,a^{m+n+1}}{(m+n+1)!} = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} a^{m+n+1}$

   [这里，$\Gamma(z)$ 为伽马函数(见附录)，以下同]

3. $\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\mathrm{d}x = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (m > 0,\ n > 0)$

4. $\int_0^a x^{q-1}(a-x)^{p-1}\mathrm{d}x = a^{p+q-1}\mathrm{B}(p,q) \quad (\mathrm{Re}\ p > 0,\ \mathrm{Re}\ q > 0)$

   [这里，$\mathrm{B}(p,q)$ 为贝塔函数(见附录)，以下同]

5. $\int_0^1 x^{q-1}(1-x)^{p-1}\mathrm{d}x = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x = \mathrm{B}(p,q) \quad (\mathrm{Re}\ p > 0,\ \mathrm{Re}\ q > 0)$

6. $\int_0^a x^\alpha (a^n - x^n)^\beta \mathrm{d}x = \dfrac{\Gamma\left(\dfrac{\alpha+1}{n}\right)\Gamma(\beta+1)}{n\Gamma\left(\dfrac{\alpha+1}{n}+\beta+1\right)} a^{\alpha+n\beta+1}$

7. $\int_a^b (x-a)^\alpha (b-x)^\beta \mathrm{d}x = \dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}(b-a)^{\alpha+\beta+1}$

8. $\int_0^\infty x^{p-\frac{1}{2}}(x+a)^{-p}(x+b)^{-p}\mathrm{d}x = \sqrt{\pi}(\sqrt{a}+\sqrt{b})^{1-2p}\dfrac{\Gamma\left(p-\dfrac{1}{2}\right)}{\Gamma(p)} \quad (\mathrm{Re}\ p > 0)$

9. $\int_a^b (x-a)^{p-1}(b-x)^{q-1}(x-c)^{-p-q}\mathrm{d}x = (b-c)^{-p}(a-c)^{-q}(b-a)^{p+q-1}\mathrm{B}(p,q)$

$(\mathrm{Re}\ p > 0,\ \mathrm{Re}\ q > 0,\ c < a < b)$

10. $\int_0^a \dfrac{x^m}{a+x}\mathrm{d}x = (-a)^m\left[\ln 2 + \sum_{k=1}^m (-1)^k \dfrac{1}{k}\right]$

11. $\int_0^\infty \dfrac{x^{p-1}}{a+x}\mathrm{d}x = \begin{cases} \pi a^{p-1}\csc p\pi & (a > 0,\ 0 < \mathrm{Re}\ p < 1) \\ -\pi(-a)^{p-1}\cot p\pi & (a < 0,\ 0 < \mathrm{Re}\ p < 1) \end{cases}$

12. $\int_0^\infty \dfrac{x^p}{a+x}\mathrm{d}x = \dfrac{\pi a^p}{\sin(p+1)\pi} \quad (0 < p < 1)$

13. $\int_0^\infty \dfrac{x^{-p}}{a+x}\mathrm{d}x = \dfrac{\pi a^{-p}}{\sin p\pi} \quad (0 < p < 1)$

14. $\int_0^a \dfrac{x^p}{(a-x)^p}\mathrm{d}x = \dfrac{ap\pi}{\sin p\pi} \quad (|\ p\ | < 1)$

15. $\int_0^\infty \dfrac{x^p}{(1+x)^3}\mathrm{d}x = \dfrac{\pi}{2}p(1-p)\csc p\pi \quad (-1 < p < 2)$

16. $\int_0^1 \dfrac{x^p}{(1-x)^p}\mathrm{d}x = p\pi\csc p\pi \quad (|\ p\ | < 1)$

17. $\int_0^1 \dfrac{x^p}{(1-x)^{p+1}}\mathrm{d}x = \int_0^1 \dfrac{(1-x)^p}{x^{p+1}}\mathrm{d}x = -\pi\csc p\pi \quad (-1 < p < 0)$

18. $\int_0^\infty \dfrac{x^{p-1}}{1+x}\mathrm{d}x = \dfrac{\pi}{\sin p\pi} \quad (0 < p < 1)$

19. $\int_0^\infty \dfrac{\mathrm{d}x}{(1+x)x^p} = \pi\csc p\pi \quad (0 < p < 1)$

20. $\int_0^\infty \dfrac{\mathrm{d}x}{(1-x)x^p} = -\pi\cot p\pi \quad (0 < p < 1)$

21. $\int_0^\infty \dfrac{x^{p-1}}{1+x^q}\mathrm{d}x = \dfrac{\pi}{q}\csc\dfrac{p\pi}{q} = \dfrac{1}{q}\mathrm{B}\left(\dfrac{p}{q},\dfrac{q-p}{q}\right) \quad (\mathrm{Re}\ q > \mathrm{Re}\ p > 0)$

22. $\int_0^\infty \dfrac{x^{p-1}}{(1+x^q)^2}\mathrm{d}x = \dfrac{(p-q)\pi}{q^2}\csc\dfrac{(p-q)\pi}{q} \quad (p < 2q)$

23. $\int_0^\infty \dfrac{x^{p-1}}{1-x^q}\mathrm{d}x = \dfrac{\pi}{q}\cot\dfrac{p\pi}{q} \quad (p < q)$

24. $\int_0^1 \dfrac{x^p}{(1-x)^{p+1}}\mathrm{d}x = \int_0^1 \dfrac{(1-x)^p}{x^{p+1}}\mathrm{d}x = -\pi\csc p\pi \quad (-1 < p < 0)$

25. $\int_0^1 \frac{x^{p-1}}{(1-x)^p(1+qx)}\mathrm{d}x = \frac{\pi}{(1+q)^p}\csc p\pi \quad (0<p<1,\ q>-1)$

26. $\int_0^\infty \frac{x^{p-1}}{(a+x)(b+x)}\mathrm{d}x = \frac{\pi}{b-a}(a^{p-1}-b^{p-1})\csc p\pi$

$(0<\mathrm{Re}\ p<2,\ |\arg a|<\pi,\ |\arg b|<\pi)$

27. $\int_0^\infty \frac{x^{p-1}}{(a+x)(b-x)}\mathrm{d}x = \frac{\pi}{b+a}(a^{p-1}\csc p\pi + b^{p-1}\cot p\pi)$

$(0<\mathrm{Re}\ p<2,\ b>0,\ |\arg a|<\pi)$

28. $\int_0^\infty \frac{x^{p-1}}{(a-x)(b-x)}\mathrm{d}x = \pi\cot p\pi\cdot\frac{a^{p-1}-b^{p-1}}{b-a} \quad (0<\mathrm{Re}\ p<2,\ a>b>0)$

29. $\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\mathrm{d}x = (1-p)\pi\csc p\pi \quad (-1<p<1)$

30. $\int_1^\infty \frac{(x-1)^{1-p}}{x^3}\mathrm{d}x = \frac{1}{2}p(1-p)\pi\csc p\pi \quad (0<p<1)$

31. $\int_a^\infty \frac{(x-a)^{p-1}}{x-b}\mathrm{d}x = (a-b)^{p-1}\pi\csc p\pi \quad (a>b,\ 0<p<1)$

32. $\int_{-\infty}^a \frac{(a-x)^{p-1}}{x-b}\mathrm{d}x = -(b-a)^{p-1}\pi\csc p\pi \quad (a<b,\ 0<p<1)$

33. $\int_0^\infty \frac{(1+x)^{p-1}}{(a+x)^{p+1}}\mathrm{d}x = \frac{1-a^{-p}}{p(a-1)} \quad (a>0)$

34. $\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\mathrm{d}x = \int_1^\infty \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\mathrm{d}x = \mathrm{B}(p,q) \quad (\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

35. $\int_0^a \frac{\left(\frac{x}{a}\right)^{\alpha-1}+\left(\frac{x}{a}\right)^{\beta-1}}{(a+x)^{\alpha+\beta}}\mathrm{d}x = \frac{\mathrm{B}(\alpha,\beta)}{a^{\alpha+\beta-1}}$

Ⅱ.1.1.2 含有 a^n+x^n，$a+bx^n$ 和 $a+2bx+cx^2$ 的积分

36. $\int_0^a \frac{\mathrm{d}x}{a+x} = \ln 2$

37. $\int_0^\infty \frac{\mathrm{d}x}{a^2+x^2} = \frac{\pi}{2a}$

38. $\int_0^\infty \frac{\mathrm{d}x}{a^3+x^3} = \frac{2\pi}{3a^2\sqrt{3}}$

39. $\int_0^\infty \frac{\mathrm{d}x}{a^n+x^n} = \frac{a\pi}{na^n\sin\frac{\pi}{n}}$

40. $\int_0^\infty \frac{x}{a^2+x^2}\mathrm{d}x = \infty$

41. $\int_0^a \frac{x}{a^2+x^2}\mathrm{d}x = \frac{1}{2}\ln 2$

42. $\int_0^\infty \frac{x}{a^3+x^3}\mathrm{d}x = \frac{2\pi}{3a\sqrt{3}}$

43. $\int_0^\infty \frac{x}{a^n+x^n}\mathrm{d}x = \frac{\pi}{na^{n-2}\sin\frac{2\pi}{n}}$

44. $\int_0^\infty \frac{x^2}{a^n+x^n}\mathrm{d}x = \frac{\pi}{na^{n-3}\sin\frac{3\pi}{n}}$

45. $\int_0^\infty \frac{x^m}{a^n+x^n}\mathrm{d}x = \frac{a^{m+1}\pi}{na^n\sin\frac{(m+1)\pi}{n}}$

46. $\int_0^a \frac{x^m}{a^n+x^n}\mathrm{d}x = a^{m-n+1}\left[\sum_{k=0}^{\infty}(-1)^k\frac{1}{m+kn+1}\right]$

47. $\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)^n} = \frac{(2n-3)!!}{(2n-2)!!}\frac{\pi}{2a^{2n-1}}$

48. $\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$

49. $\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)(a^n+x^n)} = \frac{\pi}{4a^{n+1}}$

50. $\int_0^\infty \frac{\mathrm{d}x}{(a+bx)^2} = \frac{1}{ab}$

51. $\int_0^\infty \frac{\mathrm{d}x}{(a+bx)^3} = \frac{1}{2ab^2}$

52. $\int_0^\infty \frac{\mathrm{d}x}{(a+bx)^n} = \frac{\mathrm{B}(1,n-1)}{a^{n-1}b}$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)，以下同]

53. $\int_0^\infty \frac{x^m}{(a+bx)^n}\mathrm{d}x = \frac{\mathrm{B}(m+1,n-m-1)}{a^{n-m-1}b^{m+1}}$

54. $\int_0^\infty \frac{x^m}{(a+bx)^{n+\frac{1}{2}}}\mathrm{d}x = 2^{m+1}m!\frac{(2n-2m-3)!!}{(2n-1)!!}\frac{a^{m-n+\frac{1}{2}}}{b^{m+1}}$

$(a>0,\ b>0,\ m<n-\frac{1}{2})$

55. $\int_0^\infty \frac{x^{2m}}{(a+bx^2)^n}\mathrm{d}x = \frac{(2m-1)!!(2n-2m-3)!!}{(2n-2)!!}\frac{\pi}{2a^{n-m-1}b^m\sqrt{ab}}\quad(n>m+1)$

56. $\int_0^\infty \frac{x^{2m+1}}{(a+bx^2)^n}\mathrm{d}x = \frac{m!!(n-m-2)!}{(n-1)!}\frac{1}{2a^{n-m-1}b^{m+1}}\quad(n>m+1\geqslant 1)$

57. $\int_0^a \frac{\mathrm{d}x}{a^2+ax+x^2} = \frac{\pi}{3a\sqrt{3}}$

58. $\int_0^a \frac{\mathrm{d}x}{a^2-ax+x^2} = \frac{2\pi}{3a\sqrt{3}}$

59. $\int_0^{\infty} \frac{\mathrm{d}x}{a+2bx+cx^2} = \frac{1}{\sqrt{ac-b^2}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}} \quad (ac-b^2>0)$

60. $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(a+2bx+cx^2)^n} = \frac{(2n-3)!!c^{n-1}\pi}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}} \quad (c>0,\ ac>b^2)$

61. $\int_{-\infty}^{\infty} \frac{x}{(a+2bx+cx^2)^n}\mathrm{d}x = -\frac{(2n-3)!!bc^{n-2}\pi}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}}$

$(c>0,\ ac>b^2,\ n\geqslant 2)$

Ⅱ.1.1.3 含有 $x^p \pm x^q$ 和 $1 \pm x^n$ 的积分

62. $\int_0^1 \frac{x^{p-1}-x^{-p}}{1+x}\mathrm{d}x = \pi\csc p\pi \quad (p^2<1)$

63. $\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\mathrm{d}x = \pi\cot p\pi \quad (p^2<1)$

64. $\int_0^1 \frac{x^{p}-x^{-p}}{x+1}\mathrm{d}x = \frac{1}{p}-\frac{\pi}{\sin p\pi} \quad (p^2<1)$

65. $\int_0^1 \frac{x^{p}-x^{-p}}{x-1}\mathrm{d}x = \frac{1}{p}-\frac{\pi}{\tan p\pi} \quad (p^2<1)$

66. $\int_0^1 \frac{x^{p}-x^{-p}}{1+x^2}x\mathrm{d}x = \frac{1}{2}-\frac{\pi}{2}\csc\frac{p\pi}{2} \quad (p^2<1)$

67. $\int_0^1 \frac{x^{p}-x^{-p}}{1-x^2}x\mathrm{d}x = \frac{\pi}{2}\cot\frac{p\pi}{2}-\frac{1}{p} \quad (p^2<1)$

68. $\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}\mathrm{d}x = \frac{\pi}{q}\csc\frac{p\pi}{q} \quad (q>p>0)$

69. $\int_0^1 \frac{x^{p-1}-x^{q-p-1}}{1-x^p}\mathrm{d}x = \frac{\pi}{q}\cot\frac{p\pi}{q} \quad (q>p>0)$

70. $\int_0^1 \left(\frac{1}{1-x}-\frac{px^{p-1}}{1-x^p}\right)\mathrm{d}x = \ln p$

71. $\int_0^1 \left(\frac{x^{p-1}}{1-x}-\frac{qx^{pq-1}}{1-x^q}\right)\mathrm{d}x = \ln q \quad (q>0)$

72. $\int_0^{\infty} \frac{x^{p-1}-x^{q-1}}{1-x}\mathrm{d}x = \pi(\cot p\pi-\cot q\pi) \quad (p>0,\ q>0)$

73. $\int_0^{\infty} \frac{x^{p-1}-x^{p}}{1-x^n}\mathrm{d}x = \frac{\pi}{n}\sin\frac{\pi}{n}\csc\frac{p\pi}{n}\csc\frac{(p+1)\pi}{n} \quad (0<\operatorname{Re} p<n-1)$

74. $\int_0^{\infty} \left(\frac{x^{p}-x^{-p}}{1-x}\right)^2\mathrm{d}x = 2(1-2p\pi\cot 2p\pi) \quad (0<p^2<\frac{1}{4})$

75. $\int_0^{\infty} \frac{x^{q}-1}{x(x^p-x^{-p})}\mathrm{d}x = \frac{\pi}{2p}\tan\frac{q\pi}{2p} \quad (p>q)$

Ⅱ.1.1.4 含有$\sqrt{a^n \pm x^n}$的积分

76. $\int_0^a \sqrt{a^2+x^2}\mathrm{d}x = \frac{a^2}{2}[\sqrt{2}+\ln(\sqrt{2}+1)]$

77. $\int_0^a \sqrt{a^2-x^2}\mathrm{d}x = \frac{\pi a^2}{4}$

78. $\int_0^a x\sqrt{a^2+x^2}\mathrm{d}x = \frac{a^3}{3}(2\sqrt{2}-1)$

79. $\int_0^a x\sqrt{a^2-x^2}\mathrm{d}x = \frac{a^3}{3}$

80. $\int_0^a x^{2m+1}\sqrt{a^2-x^2}\mathrm{d}x = \frac{(2m)!!}{(2m+3)!!}a^{2m+3}$

81. $\int_0^a x^{2m}\sqrt{a^2-x^2}\mathrm{d}x = \frac{(2m-1)!!}{(2m+2)!!}\frac{\pi a^{2m+2}}{2}$

82. $\int_0^a \sqrt{(a^2-x^2)^n}\mathrm{d}x = \frac{1}{2}\int_{-a}^a \sqrt{(a^2-x^2)^n}\mathrm{d}x = \frac{n!!}{(n+1)!!}\frac{\pi}{2}a^{n+1}$

$(a>0,\ n$ 为奇数$)$

83. $\int_0^a x^m\sqrt{(a^2-x^2)^n}\mathrm{d}x = \frac{1}{2}a^{m+n+1}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+n+3}{2}\right)}$

$(a>0,\ m>-1,\ n>-2)$

[这里,$\Gamma(z)$为伽马函数(见附录),以下同]

84. $\int_0^a \frac{x}{\sqrt{a-x}}\mathrm{d}x = \frac{4a\sqrt{a}}{3}$

85. $\int_0^a \frac{x^2}{\sqrt{a-x}}\mathrm{d}x = \frac{16a^2\sqrt{a}}{15}$

86. $\int_0^a \frac{x^m}{\sqrt{a-x}}\mathrm{d}x = \frac{(2m)!!}{(2m+1)!!}\frac{2a^{m+1}}{\sqrt{a}}$

87. $\int_0^a \frac{\mathrm{d}x}{\sqrt{a^2+x^2}} = \ln(\sqrt{2}+1)$

88. $\int_0^a \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$

89. $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\mathrm{d}x = (\sqrt{2}-1)a$

90. $\int_0^a \frac{x}{\sqrt{a^2-x^2}}\mathrm{d}x = a$

91. $\int_0^a \frac{x^{2m+1}}{\sqrt{a^2-x^2}}\mathrm{d}x = \frac{(2m)!!}{(2m+1)!!}a^{2m+1}$

92. $\int_0^a \frac{x^{2m}}{\sqrt{a^2-x^2}}\mathrm{d}x = \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}a^{2m}$

93. $\int_0^a \frac{\mathrm{d}x}{\sqrt{a^3-x^3}} = \frac{1.403160\cdots}{\sqrt{a}}$

94. $\int_0^a \frac{\mathrm{d}x}{\sqrt{a^4-x^4}} = \frac{5.244115\cdots}{a}$

95. $\int_0^a \frac{\mathrm{d}x}{\sqrt{a^n-x^n}} = \frac{a}{n}\sqrt{\frac{\pi}{a^n}}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

96. $\int_0^a \frac{\mathrm{d}x}{\sqrt[p]{a^n-x^n}} = \frac{a}{n\sqrt[p]{a^n}}\mathrm{B}\left(\frac{p-1}{p},\frac{1}{n}\right)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)，以下同]

97. $\int_0^a \frac{x^m}{\sqrt{a^n-x^n}}\mathrm{d}x = \frac{a^{m+1}}{n}\sqrt{\frac{\pi}{a^n}}\frac{\Gamma\left(\frac{m+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}+\frac{1}{2}\right)}$

98. $\int_0^a \frac{x^m}{\sqrt[p]{a^n-x^n}}\mathrm{d}x = \frac{a^{m+1}}{n\sqrt[p]{a^n}}\mathrm{B}\left(\frac{p-1}{p},\frac{m+1}{n}\right)$

99. $\int_0^\infty (\sqrt{a^2+x^2}-x)^n\mathrm{d}x = \frac{na^{n+1}}{n^2-1} \quad (n\geqslant 2)$

100. $\int_0^\infty \frac{\mathrm{d}x}{(x+\sqrt{a^2+x^2})^n} = \frac{n}{a^{n-1}(n^2-1)} \quad (n\geqslant 2)$

101. $\int_0^\infty \frac{x^m}{(\sqrt{a^2+x^2}+x)^n}\mathrm{d}x = \frac{n\cdot m!}{(n-m-1)(n-m+1)\cdots(m+n+1)a^{n-m+1}}$

$(a>0,\ 0\leqslant m\leqslant n-2)$

102. $\int_0^\infty \frac{\mathrm{d}x}{(1+x)\sqrt{x}} = \pi$

Ⅱ.1.2 三角函数和反三角函数的定积分

Ⅱ.1.2.1 含有 $\sin^n x$，$\cos^n x$，$\tan^n x$ 的积分，积分区间为$\left[0,\frac{\pi}{2}\right]$

103. $\int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x = 1$

104. $\int_0^{\frac{\pi}{2}} \sin^2 x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^2 x\mathrm{d}x = \frac{\pi}{4}$

105. $\int_0^{\frac{\pi}{2}} \sin^3 x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^3 x\mathrm{d}x = \frac{2}{3}$

106. $\int_0^{\frac{\pi}{2}} \sin^4 x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^4 x\mathrm{d}x = \frac{3\pi}{16}$

107. $\int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\mathrm{d}x = \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{n+2}{2}\right)}$ （n 为非负整数）

［这里，$\Gamma(z)$为伽马函数(见附录)，以下同］

108. $\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\mathrm{d}x = \frac{(2n)!!}{(2n+1)!!}$ （n 为正整数）

109. $\int_0^{\frac{\pi}{2}} \sin^{2n} x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{2n} x\mathrm{d}x = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$ （n 为正整数）

110. $\int_0^{\frac{\pi}{2}} \tan^h x\mathrm{d}x = \frac{\pi}{2\cos\frac{h\pi}{2}}$ $(0 < h < 1)$

111. $\int_0^{\frac{\pi}{2}} \sin x\cos x\mathrm{d}x = \frac{1}{2}$

112. $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\mathrm{d}x = \frac{\pi}{16}$

113. $\int_0^{\frac{\pi}{2}} \sin^3 x\cos^3 x\mathrm{d}x = \frac{1}{12}$

114. $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^4 x\mathrm{d}x = \frac{3\pi}{256}$

115. $\int_0^{\frac{\pi}{2}} \sin^{2m+1} x\cos^{2n+1} x\mathrm{d}x = \frac{m!n!}{2(m+n+1)!} = \frac{\Gamma(m+1)\Gamma(n+1)}{2\Gamma(m+n+2)}$

116. $\int_0^{\frac{\pi}{2}} \sin^{2m} x\cos^{2n} x\mathrm{d}x = \frac{\pi(2m-1)!!(2n-1)!!}{2(2m+2n)!!} = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(m+n+1)}$

117. $\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x\cos^2 x\mathrm{d}x = \frac{1}{3}$

118. $\int_0^{\frac{\pi}{2}} \sin^3 x\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x\cos^3 x\mathrm{d}x = \frac{1}{4}$

119. $\int_0^{\frac{\pi}{2}} \sin^n x\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x\cos^n x\mathrm{d}x = \frac{1}{n+1}$

120. $\int_0^{\frac{\pi}{2}} \sin^{2m+1} x\cos^{2n} x\mathrm{d}x = \frac{(2m)!!(2n-1)!!}{(2m+2n+1)!!} = \frac{\Gamma(m+1)\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma\left(m+n+\frac{3}{2}\right)}$

121. $\int_0^{\frac{\pi}{2}} \sin^{2m} x\cos^{2n+1} x\mathrm{d}x = \frac{(2n)!!(2m-1)!!}{(2m+2n+1)!!} = \frac{\Gamma(n+1)\Gamma\left(m+\frac{1}{2}\right)}{2\Gamma\left(m+n+\frac{3}{2}\right)}$

122. $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x \mathrm{d}x = \frac{1}{2}\mathrm{B}\left(\frac{m}{2}, \frac{n}{2}\right)$ （m 和 n 都是正整数）

[这里，$\mathrm{B}(p,q)$ 为贝塔函数（见附录）]

Ⅱ.1.2.2 含有 $\sin^n x$，$\cos^n x$，$\tan^n x$ 的积分，积分区间为 $[0,\pi]$

123. $\int_0^{\pi} \sin x \mathrm{d}x = 2$

124. $\int_0^{\pi} \cos x \mathrm{d}x = 0$

125. $\int_0^{\pi} \sin^2 x \mathrm{d}x = \int_0^{\pi} \cos^2 x \mathrm{d}x = \frac{\pi}{2}$

126. $\int_0^{\pi} \sin^{2m+1} x \mathrm{d}x = \frac{2(2m)!!}{(2m+1)!!}$

127. $\int_0^{\pi} \cos^{2m+1} x \mathrm{d}x = 0$

128. $\int_0^{\pi} \sin^{2m} x \mathrm{d}x = \int_0^{\pi} \cos^{2m} x \mathrm{d}x = \frac{\pi(2m-1)!!}{(2m)!!}$

129. $\int_0^{\pi} \sin x \cos x \mathrm{d}x = 0$

130. $\int_0^{\pi} \sin^2 x \cos^2 x \mathrm{d}x = \frac{\pi}{8}$

131. $\int_0^{\pi} \sin^3 x \cos^3 x \mathrm{d}x = 0$

132. $\int_0^{\pi} \sin^4 x \cos^4 x \mathrm{d}x = \frac{3\pi}{128}$

133. $\int_0^{\pi} \sin^{2m+1} x \cos^{2m+1} x \mathrm{d}x = 0$

134. $\int_0^{\pi} \sin^{2m} x \cos^{2m} x \mathrm{d}x = \mathrm{B}\left(m+\frac{1}{2}, m+\frac{1}{2}\right)$

[这里，$\mathrm{B}(p,q)$ 为贝塔函数（见附录）]

135. $\int_0^{\pi} x \sin x \mathrm{d}x = \pi$

136. $\int_0^{\pi} x \cos x \mathrm{d}x = -2$

137. $\int_0^{\pi} x \sin^2 x \mathrm{d}x = \int_0^{\pi} x \cos^2 x \mathrm{d}x = \frac{\pi^2}{4}$

138. $\int_0^{\pi} x \sin^{2n} x \mathrm{d}x = \int_0^{\pi} x \cos^{2n} x \mathrm{d}x = \frac{\pi^2(2n-1)!!}{2(2n)!!}$

139. $\int_0^{\pi} x \sin^{2n+1} x \mathrm{d}x = \frac{\pi(2n)!!}{(2n+1)!!}$

140. $\int_0^{\pi} x\cos^{2n+1}x\mathrm{d}x = -\frac{2}{4^n}\sum_{k=0}^{n}\binom{2n+1}{k}\frac{1}{(2n-2k-1)^2}$

141. $\int_0^{\pi} x\tan x\mathrm{d}x = -\pi\ln 2$

Ⅱ.1.2.3 含有 $\sin nx$ 和 $\cos nx$ 的积分，积分区间为$[0,\pi]$

142. $\int_0^{\pi}\sin nx\mathrm{d}x = \frac{1-(-1)^n}{n}$

143. $\int_0^{\pi} x\sin nx\mathrm{d}x = -\frac{(-1)^n\pi}{n}$

144. $\int_0^{\pi} x^2\sin nx\mathrm{d}x = \frac{2[(-1)^n-1]}{n^3} - \frac{(-1)^n\pi^2}{n}$

145. $\int_0^{\pi} x^3\sin nx\mathrm{d}x = \frac{6(-1)^n\pi}{n^3} - \frac{(-1)^n\pi^3}{n}$

146. $\int_0^{\pi}\cos nx\mathrm{d}x = 0$

147. $\int_0^{\pi} x\cos nx\mathrm{d}x = \frac{(-1)^n-1}{n^2}$

148. $\int_0^{\pi} x^2\cos nx\mathrm{d}x = \frac{(-1)^n 2\pi}{n^2}$

149. $\int_0^{\pi} x^3\cos nx\mathrm{d}x = \frac{(-1)^n 3\pi^2}{n^2} - \frac{6[(-1)^n-1]}{n^4}$

150. $\int_0^{\pi}\sin ax\sin nx\mathrm{d}x = \frac{(-1)^n n\sin a\pi}{a^2-n^2}$

151. $\int_0^{\pi}\cos ax\sin nx\mathrm{d}x = \frac{n[(-1)^n\cos a\pi-1]}{a^2-n^2}$

152. $\int_0^{\pi}\sin ax\cos nx\mathrm{d}x = \frac{a[1-(-1)^n\cos a\pi]}{a^2-n^2}$

153. $\int_0^{\pi}\cos ax\cos nx\mathrm{d}x = \frac{(-1)^n a\sin a\pi}{a^2-n^2}$

154. $\int_0^{\pi}\sin^2 mx\mathrm{d}x = \int_0^{\pi}\cos^2 mx\mathrm{d}x = \frac{\pi}{2}$ （m 为整数，$m\neq 0$）

155. $\int_0^{\pi}\sin mx\sin nx\mathrm{d}x = \int_0^{\pi}\cos mx\cos nx\mathrm{d}x = 0$ （$m\neq n$，m 和 n 都为整数）

156. $\int_0^{\pi}\sin nx\cos nx\mathrm{d}x = \int_0^{\frac{\pi}{n}}\sin nx\cos nx\mathrm{d}x = 0$ （n 为整数）

157. $\int_0^{\pi}\sin ax\cos bx\mathrm{d}x = \begin{cases}\frac{2a}{a^2-b^2} & (a-b\text{ 为奇数})\\ 0 & (a-b\text{ 为偶数})\end{cases}$

Ⅱ.1.2.4 含有 $\sin nx$ 和 $\cos nx$ 的积分，积分区间为$[-\pi,\pi]$

158. $\int_{-\pi}^{\pi}\sin nx\mathrm{d}x=0$

159. $\int_{-\pi}^{\pi}x\sin nx\mathrm{d}x=-\frac{(-1)^n2\pi}{n}$

160. $\int_{-\pi}^{\pi}x^2\sin nx\mathrm{d}x=0$

161. $\int_{-\pi}^{\pi}x^3\sin nx\mathrm{d}x=\frac{(-1)^n12\pi}{n^3}-\frac{(-1)^n2\pi^3}{n}$

162. $\int_{-\pi}^{\pi}\cos nx\mathrm{d}x=0$

163. $\int_{-\pi}^{\pi}x\cos nx\mathrm{d}x=0$

164. $\int_{-\pi}^{\pi}x^2\cos nx\mathrm{d}x=\frac{(-1)^n4\pi}{n^2}$

165. $\int_{-\pi}^{\pi}x^3\cos nx\mathrm{d}x=0$

166. $\int_{-\pi}^{\pi}\sin ax\sin nx\mathrm{d}x=\frac{(-1)^n2n\sin a\pi}{a^2-n^2}$

167. $\int_{-\pi}^{\pi}\cos ax\sin nx\mathrm{d}x=0$

168. $\int_{-\pi}^{\pi}\sin ax\cos nx\mathrm{d}x=0$

169. $\int_{-\pi}^{\pi}\cos ax\cos nx\mathrm{d}x=\frac{(-1)^n2a\sin a\pi}{a^2-n^2}$

Ⅱ.1.2.5 含有其他倍角三角函数的积分

170. $\int_0^{\frac{\pi}{2}}\frac{\sin(2n-1)x}{\sin x}\mathrm{d}x=\frac{\pi}{2}$

171. $\int_0^{\frac{\pi}{2}}\frac{\sin 2nx}{\sin nx}\mathrm{d}x=2\left[1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^{n-1}}{2n-1}\right]$

172. $\int_0^{\frac{\pi}{2}}\frac{\sin 2nx\cos x}{\sin x}\mathrm{d}x=\frac{\pi}{2}$

173. $\int_0^{\frac{\pi}{2}}\frac{\cos 2nx}{1-a^2\sin^2x}\mathrm{d}x=\frac{(-1)^n\pi}{2\sqrt{1-a^2}}\left(\frac{1-\sqrt{1-a^2}}{a}\right)^{2n}\quad(a^2<1)$

174. $\int_0^{\frac{\pi}{2}}\frac{\cos 2nx}{(a^2\cos^2x+b^2\sin^2x)^{n+1}}\mathrm{d}x=\binom{2n}{n}\frac{(b^2-a^2)^n}{(2ab)^{2n+1}}\pi\quad(a>0,\ b>0)$

175. $\int_0^{\frac{\pi}{2}} \sin ax \sin nx \mathrm{d}x = \frac{1}{a^2 - n^2}\left(n \sin \frac{a\pi}{2} \cos \frac{n\pi}{2} - a \cos \frac{a\pi}{2} \sin \frac{n\pi}{2}\right)$

176. $\int_0^{\frac{\pi}{2}} \cos ax \sin nx \mathrm{d}x = -\frac{1}{a^2 - n^2}\left(n - n \cos \frac{a\pi}{2} \cos \frac{n\pi}{2} - a \sin \frac{a\pi}{2} \sin \frac{n\pi}{2}\right)$

177. $\int_0^{\frac{\pi}{2}} \cos ax \cos nx \mathrm{d}x = \frac{1}{a^2 - n^2}\left(a \sin \frac{a\pi}{2} \cos \frac{n\pi}{2} - n \cos \frac{a\pi}{2} \sin \frac{n\pi}{2}\right)$

178. $\int_0^{\pi} \frac{\sin nx \cos mx}{\sin x} \mathrm{d}x = \begin{cases} 0 & (n \leqslant m) \\ \pi & (n > m,\ m + n \text{ 为奇数}) \\ 0 & (n > m,\ m + n \text{ 为偶数}) \end{cases}$

179. $\int_0^{\pi} \frac{\sin nx}{\sin x} \mathrm{d}x = \begin{cases} 0 & (n \text{ 为偶数}) \\ \pi & (n \text{ 为奇数}) \end{cases}$

180. $\int_0^{\pi} \frac{\sin 2nx}{\cos x} \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\cos x} \mathrm{d}x$

$$= (-1)^{n-1} 4\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n - 1}\right]$$

181. $\int_0^{\pi} \frac{\cos(2n + 1)x}{\cos x} \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \frac{\cos(2n + 1)x}{\cos x} \mathrm{d}x = (-1)^n \pi$

182. $\int_0^{\pi} \frac{x \sin(2n + 1)x}{\sin x} \mathrm{d}x = \frac{\pi^2}{2} \quad (n = 0, 1, 2, \cdots)$

183. $\int_0^{\pi} \frac{x \sin 2nx}{\sin x} \mathrm{d}x = 4\sum_{k=0}^{\infty} (2k + 1)^{-2} \quad (n = 1, 2, \cdots)$

184. $\int_0^{\pi} \frac{\cos nx}{1 + a \cos x} \mathrm{d}x = \frac{\pi}{\sqrt{1 - a^2}}\left(\frac{\sqrt{1 - a^2} - 1}{a}\right)^n \quad (a^2 < 1)$

185. $\int_0^{\pi} (1 - 2a \cos x + a^2)^n \mathrm{d}x = \pi \sum_{k=0}^{n} \binom{n}{k}^2 a^{2k}$

186. $\int_0^{\pi} (1 - 2a \cos x + a^2)^n \cos nx \mathrm{d}x = (-1)^n \pi a^n$

187. $\int_0^{2\pi} (1 - \cos x)^n \sin nx \mathrm{d}x = 0$

188. $\int_0^{2\pi} (1 - \cos x)^n \cos nx \mathrm{d}x = (-1)^n \frac{\pi}{2^{n-1}}$

Ⅱ.1.2.6 含有三角函数的代数式的积分，积分区间为 $\left[0, \frac{\pi}{2}\right]$

189. $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x} \mathrm{d}x = 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right) = 2G$

[这里，G 为卡塔兰常数(见附录)，以下同]

190. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1 + \sin x} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1 + \cos x} = 1$

191. $\int_0^{\frac{\pi}{2}} \frac{x}{1+\sin x}\mathrm{d}x = \ln 2$

192. $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x}\mathrm{d}x = \frac{\pi}{2} - \ln 2$

193. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x}\mathrm{d}x = \frac{\pi}{2} - 1$

194. $\int_0^{\frac{\pi}{2}} \frac{x\sin x}{1+\cos x}\mathrm{d}x = -\frac{\pi}{2}\ln 2 + 2G$

195. $\int_0^{\frac{\pi}{2}} \frac{x\sin x}{1-\cos x}\mathrm{d}x = \frac{\pi}{2}\ln 2 + 2G$

196. $\int_0^{\frac{\pi}{2}} \frac{x\cos x}{1+\sin x}\mathrm{d}x = \pi\ln 2 - 2G$

197. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+a\sin x} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+a\cos x} = \frac{\arccos a}{\sqrt{1-a^2}} \quad (|a|<1)$

198. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(1\pm a\sin x)^2} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(1\pm a\cos x)^2} = \frac{\pi \mp 2\arcsin a}{2\sqrt{(1-a^2)^3}} \mp \frac{a}{1-a^2}$

$\left(0 < \arcsin a < \frac{\pi}{2}\right)$

199. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(1\pm a^2\sin^2 x)^2} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(1\pm a^2\cos^2 x)^2} = \frac{(2\pm a^2)\pi}{4\sqrt{(1\pm a^2)^3}} \quad (|a|<1)$

200. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sin x \pm \cos x} = \mp\frac{1}{\sqrt{2}}\ln\left(\tan\frac{\pi}{8}\right)$

201. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(\sin x \pm \cos x)^2} = \pm 1$

202. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(a\sin x + b\cos x)^2} = \frac{1}{ab}$

203. $\int_0^{\frac{\pi}{2}} \frac{x}{(a\sin x + b\cos x)^2}\mathrm{d}x = \frac{ab}{a^2+b^2}\frac{\pi}{2} - \frac{\ln ab}{a^2+b^2}$

204. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2|ab|}$

205. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{(a^2+b^2)\pi}{4a^3b^3} \quad (a>0,\ b>0)$

206. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{a^2\sin^2 x + b^2\cos^2 x}\mathrm{d}x = \frac{\pi}{2a(a+b)}$

207. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{a^2\sin^2 x + b^2\cos^2 x}\mathrm{d}x = \frac{\pi}{2b(a+b)}$

208. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\mathrm{d}x = \frac{\pi}{4a^3b}$

209. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\mathrm{d}x = \frac{\pi}{4ab^3}$

210. $\int_0^{\frac{\pi}{2}} x\tan x\mathrm{d}x = \infty$

211. $\int_0^{\frac{\pi}{2}} x\cot x\mathrm{d}x = \frac{\pi}{2}\ln 2$

212. $\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\tan x\mathrm{d}x = \frac{1}{2}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\tan x\mathrm{d}x = \frac{\pi}{2}\ln 2$

213. $\int_0^{\frac{\pi}{2}} \frac{x\cot x}{\cos 2x}\mathrm{d}x = \frac{\pi}{4}\ln 2$

214. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1 + \tan^m x} = \frac{\pi}{4}$ （m 为非负整数）

215. $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\mathrm{d}x = \frac{(2\pi)^{\frac{3}{2}}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}$

[这里，$\Gamma(z)$为伽马函数(见附录)]

Ⅱ.1.2.7 含有三角函数的代数式的积分，积分区间为$[0,\pi]$

216. $\int_0^{\pi} \frac{\mathrm{d}x}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}$ $(a > b \geqslant 0)$

217. $\int_0^{\pi} \frac{\mathrm{d}x}{1 \pm b\sin x} = \frac{\pi \mp 2\arcsin b}{\sqrt{1 - b^2}}$

218. $\int_0^{\pi} \frac{\mathrm{d}x}{1 \pm b\cos x} = \frac{\pi}{\sqrt{1 - b^2}}$ $(|b| < 1)$

219. $\int_0^{\pi} \frac{\mathrm{d}x}{(1 \pm b\sin x)^2} = \frac{\pi \mp 2\arcsin b}{\sqrt{(1 - b^2)^3}} \mp \frac{2b}{1 - b^2}$

220. $\int_0^{\pi} \frac{\mathrm{d}x}{(1 \pm b\cos x)^2} = \frac{\pi}{\sqrt{(1 - b^2)^3}}$

221. $\int_0^{\pi} \frac{\mathrm{d}x}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi^2}{2ab}$

222. $\int_0^{\pi} \frac{x\sin x\cos x}{a^2\sin^2 x - b^2\cos^2 x}\mathrm{d}x = \frac{\pi}{b^2 - a^2}\ln\frac{a + b}{2b}$

Ⅱ.1.2.8 三角函数的幂函数的积分

223. $\int_0^{\frac{\pi}{2}} \sin^{p-1} x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{p-1} x\mathrm{d}x = 2^{p-2}\mathrm{B}\left(\frac{p}{2}, \frac{p}{2}\right)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)，以下同]

224. $\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}} x\mathrm{d}x = \frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

225. $\int_0^{\frac{\pi}{2}} \sin^{p-1}x\cos^{q-1}x\mathrm{d}x = \frac{1}{2}\mathrm{B}\left(\frac{p}{2},\frac{q}{2}\right)$ $(\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

226. $\int_0^{\frac{\pi}{2}} \tan^{\pm p}x\mathrm{d}x = \frac{\pi}{2}\sec\frac{p\pi}{2}$ $(|\ \mathrm{Re}\ p\ |<1)$

227. $\int_0^{\frac{\pi}{2}} \tan^{p-1}x\cos^{2q-2}x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cot^{p-1}x\sin^{2q-2}x\mathrm{d}x = \frac{1}{2}\mathrm{B}\left(\frac{p}{2},q-\frac{p}{2}\right)$

$(0<\mathrm{Re}\ p<2\mathrm{Re}\ q)$

228. $\int_0^{\frac{\pi}{2}} \frac{\tan^p x}{\cos^p x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cot^p x}{\sin^p x}\mathrm{d}x = \frac{\Gamma(p)\Gamma\left(\frac{1}{2}-p\right)}{2^p\sqrt{\pi}}\sin\frac{p\pi}{2}$

$\left(-1<\mathrm{Re}\ p<\frac{1}{2}\right)$

229. $\int_0^{\frac{\pi}{4}} \frac{\sin^p x}{\cos^{p+2}x}\mathrm{d}x = \frac{1}{p+1}$ $(p>-1)$

Ⅱ.1.2.9 三角函数的幂函数与线性函数的三角函数组合的积分

230. $\int_0^{\pi} \sin^n x\sin 2mx\mathrm{d}x = 0$

231. $\int_0^{\pi} \sin^p x\sin px\mathrm{d}x = 2^{-p}\pi\sin\frac{p\pi}{2}$ $(\mathrm{Re}\ p>-1)$

232. $\int_0^{\pi} \sin^p x\cos px\mathrm{d}x = \frac{\pi}{2^p}\cos\frac{p\pi}{2}$ $(\mathrm{Re}\ p>-1)$

233. $\int_0^{\pi} \sin^{p-1}x\sin ax\mathrm{d}x = \frac{\pi\sin\frac{a\pi}{2}}{2^{p-1}p\mathrm{B}\left(\frac{p+a+1}{2},\frac{p-a+1}{2}\right)}$ $(\mathrm{Re}\ p>0)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)，以下同]

234. $\int_0^{\pi} \sin^{p-1}x\cos ax\mathrm{d}x = \frac{\pi\cos\frac{a\pi}{2}}{2^{p-1}p\mathrm{B}\left(\frac{p+a+1}{2},\frac{p-a+1}{2}\right)}$ $(\mathrm{Re}\ p>0)$

235. $\int_0^{\frac{\pi}{2}} \cos^{p-1}x\cos ax\mathrm{d}x = \frac{\pi}{2^p p\mathrm{B}\left(\frac{p+a+1}{2},\frac{p-a+1}{2}\right)}$ $(\mathrm{Re}\ p>0)$

236. $\int_0^{\frac{\pi}{2}} \cos^n x\cos nx\mathrm{d}x = \frac{\pi}{2^{n+1}}$

237. $\int_0^{\frac{\pi}{2}} \tan^{\pm p}x\sin 2x\mathrm{d}x = \frac{p\pi}{2}\csc\frac{p\pi}{2}$ $(0<\mathrm{Re}\ p<2)$

238. $\int_0^{\frac{\pi}{2}} \tan^{\pm p}x\cos 2x\mathrm{d}x = \mp\frac{p\pi}{2}\sec\frac{p\pi}{2}$ $(|\ \mathrm{Re}\ p\ |<1)$

239. $\int_0^{\frac{\pi}{2}} \frac{\tan^{2p}x}{\cos x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cot^{2p}x}{\sin x}\mathrm{d}x = \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma(-p)}{2\sqrt{\pi}} \quad \left(-\frac{1}{2} < \mathrm{Re}\ p < 1\right)$

[这里，$\Gamma(z)$为伽马函数(见附录)]

240. $\int_0^{\frac{\pi}{2}} \frac{\cos^{p-1}x\sin px}{\sin x}\mathrm{d}x = \frac{\pi}{2} \quad (p > 0)$

Ⅱ.1.2.10 三角函数的幂函数与三角函数的有理函数组合的积分

241. $\int_0^{\pi} \frac{\sin^m x}{1+\cos x}\mathrm{d}x = 2^{m-1}\mathrm{B}\left(\frac{m-1}{2},\frac{m+1}{2}\right) \quad (m \geqslant 2)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)，以下同]

242. $\int_0^{\pi} \frac{\sin^m x}{1-\cos x}\mathrm{d}x = 2^{m-1}\mathrm{B}\left(\frac{m-1}{2},\frac{m+1}{2}\right) \quad (m \geqslant 2)$

243. $\int_0^{\pi} \frac{\sin^2 x}{p+q\cos x}\mathrm{d}x = \frac{p\pi}{q^2}\left(1-\sqrt{1-\frac{q^2}{p^2}}\right)$

244. $\int_0^{\pi} \frac{\sin^3 x}{p+q\cos x}\mathrm{d}x = \frac{2p}{q^2} + \frac{1}{q}\left(1-\frac{p^2}{q^2}\right)\ln\frac{p+q}{p-q}$

245. $\int_0^{\pi} \frac{\tan^{\pm p}x}{1+\cos t\sin 2x}\mathrm{d}x = \pi\csc t\sin pt\sec p\pi \quad (|\mathrm{Re}\ p| < 1,\ t^2 < \pi^2)$

246. $\int_0^{\frac{\pi}{2}} \frac{\sin^{p-1}x\cos^{-p}x}{a\cos x+b\sin x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin^{-p}x\cos^{p-1}x}{a\sin x+b\cos x}\mathrm{d}x = \frac{\pi\csc p\pi}{a^{1-p}b^p} \quad (ab > 0,\ 0 < p < 1)$

247. $\int_0^{\frac{\pi}{2}} \frac{\cos^p x\cos px}{a^2\sin^2 x+b^2\cos^2 x}\mathrm{d}x = \frac{\pi a^{p-1}}{2b(a+b)^p} \quad (a > 0,\ b > 0,\ p > -1)$

248. $\int_0^{\frac{\pi}{2}} \frac{\sin^{2p-1}x\cos^{2q-1}x}{(a^2\sin^2 x+b^2\cos^2 x)^{p+q}}\mathrm{d}x = \frac{1}{2a^{2p}b^{2q}}\mathrm{B}(p,q) \quad (\mathrm{Re}\ p > 0,\ \mathrm{Re}\ q > 0)$

249. $\int_0^{\frac{\pi}{2}} \frac{\sin^{n-1}x\cos^{n-1}x}{(a^2\cos^2 x+b^2\sin^2 x)^n}\mathrm{d}x = \frac{\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)}{2(ab)^n} \quad (ab > 0)$

250. $\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}x}{(a^2\cos^2 x+b^2\sin^2 x)^{n+1}}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^{2n}x}{(a^2\sin^2 x+b^2\cos^2 x)^{n+1}}\mathrm{d}x$

$= \frac{(2n-1)!!\pi}{2^{n+1}n!(ab)^{2n+1}} \quad (ab > 0)$

251. $\int_0^{\frac{\pi}{2}} \frac{\tan^p x}{(\sin x+\cos x)\sin x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cot^p x}{(\cos x+\sin x)\cos x}\mathrm{d}x = \pi\csc p\pi$

$(0 < \mathrm{Re}\ p < 1)$

252. $\int_0^{\frac{\pi}{2}} \frac{\tan^p x}{(\sin x-\cos x)\sin x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cot^p x}{(\cos x-\sin x)\cos x}\mathrm{d}x = -\pi\cot p\pi$

$(0 < \mathrm{Re}\ p < 1)$

253. $\int_0^{\frac{\pi}{2}} \frac{\cot^{p+\frac{1}{2}}x}{(\sin x+\cos x)\cos x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\tan^{p-\frac{1}{2}}x}{(\sin x+\cos x)\cos x}\mathrm{d}x = \pi\sec p\pi$

$\left(|\operatorname{Re} p|<\frac{1}{2}\right)$

254. $\int_0^{\frac{\pi}{2}} \frac{\tan^{1-2p}x}{a^2\cos^2 x+b^2\sin^2 x}\mathrm{d}x=\int_0^{\frac{\pi}{2}} \frac{\cot^{1-2p}x}{a^2\sin^2 x+b^2\cos^2 x}\mathrm{d}x=\frac{\pi}{2a^{2p}b^{2-2p}\sin p\pi}$

$(0<\operatorname{Re} p<1)$

255. $\int_0^{\frac{\pi}{2}} \frac{\tan^p x}{1-a\sin^2 x}\mathrm{d}x=\int_0^{\frac{\pi}{2}} \frac{\cot^p x}{1-a\cos^2 x}\mathrm{d}x=\frac{\pi\sec\frac{p\pi}{2}}{2\sqrt{(1-a)^{p+1}}}\quad(|\operatorname{Re} p|<1,\ a<1)$

256. $\int_0^{\frac{\pi}{2}} \frac{\tan^{\pm p}x}{(\sin x+\cos x)^2}\mathrm{d}x=\frac{p\pi}{\sin p\pi}\quad(0<\operatorname{Re} p<1)$

257. $\int_0^{\frac{\pi}{2}} \frac{\tan^{\pm(p-1)}x}{\cos^2 x-\sin^2 x}\mathrm{d}x=\pm\frac{\pi}{2}\cot\frac{p\pi}{2}\quad(0<\operatorname{Re} p<2)$

Ⅱ.1.2.11 含有三角函数的线性函数的幂函数的积分

258. $\int_0^{\frac{\pi}{2}}(\sec x-1)^p\sin x\mathrm{d}x=\int_0^{\frac{\pi}{2}}(\csc x-1)^p\cos x\mathrm{d}x=p\pi\csc p\pi\quad(|\operatorname{Re} p|<1)$

259. $\int_0^{\frac{\pi}{2}}(\sec x-1)^p\tan x\mathrm{d}x=\int_0^{\frac{\pi}{2}}(\csc x-1)^p\cot x\mathrm{d}x=-\pi\csc p\pi$

$(-1<\operatorname{Re} p<0)$

260. $\int_0^{\frac{\pi}{2}}(\csc x-1)^p\sin 2x\mathrm{d}x=(1-p)p\pi\csc p\pi\quad(-1<\operatorname{Re} p<2)$

261. $\int_0^{\frac{\pi}{4}}\frac{(\cot x-1)^p}{\sin 2x}\mathrm{d}x=-\frac{\pi}{2}\csc p\pi\quad(-1<\operatorname{Re} p<0)$

262. $\int_0^{\frac{\pi}{4}}\frac{(\cot x-1)^p}{\cos^2 x}\mathrm{d}x=p\pi\csc p\pi\quad(|\operatorname{Re} p|<1)$

263. $\int_0^{\frac{\pi}{4}}\frac{(\cos x-\sin x)^p}{\sin^p x\sin 2x}\mathrm{d}x=-\frac{\pi}{2}\csc p\pi\quad(-1<\operatorname{Re} p<0)$

264. $\int_0^{\frac{\pi}{2}}\frac{\sin^{p-1}x\cos^{q-1}x}{(\sin x+\cos x)^{p+q}}\mathrm{d}x=\mathrm{B}(p,q)\quad(\operatorname{Re} p>0,\ \operatorname{Re} q>0)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)]

265. $\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sqrt{1+p^2\sin^2 x}}\mathrm{d}x=\frac{1}{p}\arctan p\quad(p\neq 0)$

266. $\int_0^{\frac{\pi}{4}}\frac{\sqrt{\sec 2x}-1}{\tan x}\mathrm{d}x=\ln 2$

267. $\int_0^{u}\sqrt{\frac{\cos 2x-\cos 2u}{\cos 2x+1}}\mathrm{d}x=\frac{\pi}{2}(1-\cos u)\quad\left(u^2<\frac{\pi^2}{4}\right)$

Ⅱ.1.2.12 含有其他形式的三角函数的幂函数的积分

268. $\int_0^{\frac{\pi}{2}} \frac{\sin^{2p-1}x\cos^{2q-1}x}{(1-k^2\sin^2x)^{p+q}}\mathrm{d}x = \frac{\mathrm{B}(p,q)}{2(1-k^2)^p}$ $(\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)]

269. $\int_0^{\frac{\pi}{2}} \frac{\sin^p x-\csc^p x}{\cos x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^p x-\sec^p x}{\sin x}\mathrm{d}x = -\frac{\pi}{2}\tan\frac{p\pi}{2}$ $(|\ \mathrm{Re}\ p\ |<1)$

270. $\int_0^{\frac{\pi}{2}} \frac{\sin^p x+\sin^q x}{\sin^{p+q}x+1}\cot x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^p x+\cos^q x}{\cos^{p+q}x+1}\tan x\mathrm{d}x$

$= \frac{\pi}{p+q}\sec\left(\frac{p-q}{p+q}\cdot\frac{\pi}{2}\right)$ $(\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

271. $\int_0^{\frac{\pi}{2}} \frac{\sin^p x-\sin^q x}{\sin^{p+q}x-1}\cot x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^p x-\cos^q x}{\cos^{p+q}x-1}\tan x\mathrm{d}x$

$= \frac{\pi}{p+q}\tan\left(\frac{p-q}{p+q}\cdot\frac{\pi}{2}\right)$ $(\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

272. $\int_0^{\frac{\pi}{2}} \frac{\tan x}{\cos^p x+\sec^p x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cot x}{\sin^p x+\csc^p x}\mathrm{d}x = \frac{\pi}{4p}$

273. $\int_0^{\frac{\pi}{4}} \frac{\tan^{p-1}x-\cot^{p-1}x}{\cos 2x}\mathrm{d}x = \frac{\pi}{2}\cot\frac{p\pi}{2}$ $(|\ \mathrm{Re}\ p\ |<2)$

274. $\int_0^{\frac{\pi}{4}} \frac{\tan^p x-\cot^p x}{\cos 2x}\tan x\mathrm{d}x = -\frac{1}{p}+\frac{\pi}{2}\cot\frac{p\pi}{2}$ $(-2<\mathrm{Re}\ p<0)$

275. $\int_0^{\frac{\pi}{4}} \frac{\tan^p x+\cot^p x}{1+\cos t\sin 2x}\mathrm{d}x = \pi\csc t\csc p\pi\sin pt$ $(t\neq n\pi,\ |\ \mathrm{Re}\ p\ |<1)$

276. $\int_0^{\frac{\pi}{4}} \frac{\tan^{p-1}x+\cot^p x}{(\sin x+\cos x)\cos x}\mathrm{d}x = \pi\csc p\pi$ $(0<\mathrm{Re}\ p<1)$

277. $\int_0^{\frac{\pi}{4}} \frac{\tan^{p-1}x-\cot^p x}{(\cos x-\sin x)\cos x}\mathrm{d}x = \pi\cot p\pi$ $(0<\mathrm{Re}\ p<1)$

278. $\int_0^{\frac{\pi}{4}} \frac{\tan^p x-\cot^p x}{(\sin x+\cos x)\cos x}\mathrm{d}x = \frac{1}{p}-\pi\csc p\pi$ $(0<\mathrm{Re}\ p<1)$

279. $\int_0^{\frac{\pi}{4}} \frac{\tan^p x-\cot^p x}{(\cos x-\sin x)\cos x}\mathrm{d}x = -\frac{1}{p}+\pi\cot p\pi$ $(0<\mathrm{Re}\ p<1)$

280. $\int_0^{\frac{\pi}{4}} (\sin^p 2x-\csc^p 2x)\tan\left(\frac{\pi}{4}+x\right)\mathrm{d}x = \int_0^{\frac{\pi}{4}} (\cos^p 2x-\sec^p 2x)\cot x\mathrm{d}x$

$= -\frac{1}{2p}+\frac{\pi}{2}\cot p\pi$ $(|\ \mathrm{Re}\ p\ |<1)$

281. $\int_0^{\frac{\pi}{4}} (\sin^p 2x-\csc^p 2x)\cot\left(\frac{\pi}{4}+x\right)\mathrm{d}x = \int_0^{\frac{\pi}{4}} (\cos^p 2x-\sec^p 2x)\tan x\mathrm{d}x$

$= \frac{1}{2p}-\frac{\pi}{2}\csc p\pi$ $(|\ \mathrm{Re}\ p\ |<1)$

282. $\int_0^{\frac{\pi}{4}}(\sin^{p-1}2x-\csc^p 2x)\tan\left(\frac{\pi}{4}+x\right)\mathrm{d}x=\int_0^{\frac{\pi}{4}}(\cos^{p-1}2x-\sec^p 2x)\cot x\mathrm{d}x$

$=\frac{\pi}{2}\cot p\pi\quad(0<\mathrm{Re}\ p<1)$

283. $\int_0^{\frac{\pi}{4}}(\sin^{p-1}2x+\csc^p 2x)\cot\left(\frac{\pi}{4}+x\right)\mathrm{d}x=\int_0^{\frac{\pi}{4}}(\cos^{p-1}2x+\sec^p 2x)\tan x\mathrm{d}x$

$=\frac{\pi}{2}\csc p\pi\quad(0<\mathrm{Re}\ p<1)$

284. $\int_0^{\frac{\pi}{4}}(\tan^p x+\cot^p x)\mathrm{d}x=\frac{\pi}{2}\sec\frac{p\pi}{2}\quad(|\mathrm{Re}\ p|<1)$

285. $\int_0^{\frac{\pi}{4}}(\tan^p x-\cot^p x)\tan x\mathrm{d}x=\frac{1}{p}-\frac{\pi}{2}\csc\frac{p\pi}{2}\quad(0<\mathrm{Re}\ p<2)$

286. $\int_0^{\frac{\pi}{4}}(\tan^p x+\cot^p x)(\tan^q x+\cot^q x)\mathrm{d}x=\frac{2\pi\cos\frac{p\pi}{2}\cos\frac{q\pi}{2}}{\cos p\pi+\cos q\pi}$

$(|\mathrm{Re}\ p|<1,\ |\mathrm{Re}\ q|<1)$

287. $\int_0^{\frac{\pi}{4}}(\tan^p x-\cot^p x)(\tan^q x-\cot^q x)\mathrm{d}x=\frac{2\pi\sin\frac{p\pi}{2}\sin\frac{q\pi}{2}}{\cos p\pi+\cos q\pi}$

$(|\mathrm{Re}\ p|<1,\ |\mathrm{Re}\ q|<1)$

288. $\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}x}{(\tan^p x+\cot^p x)\sin 2x}=\frac{\pi}{8p}\quad(\mathrm{Re}\ p\neq 0)$

Ⅱ.1.2.13 更复杂自变数的三角函数的积分

289. $\int_0^{\infty}\sin ax^p\mathrm{d}x=\frac{1}{pa^{\frac{1}{p}}}\Gamma\left(\frac{1}{p}\right)\sin\frac{\pi}{2p}\quad(a>0,\ p>1)$

[这里,$\Gamma(z)$为伽马函数(见附录),以下同]

290. $\int_0^{\infty}\cos ax^p\mathrm{d}x=\frac{1}{pa^{\frac{1}{p}}}\Gamma\left(\frac{1}{p}\right)\cos\frac{\pi}{2p}\quad(a>0,\ p>1)$

291. $\int_0^{\infty}\sin ax^2\mathrm{d}x=\int_0^{\infty}\cos ax^2\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{2a}}\quad(a>0)$

292. $\int_0^{\infty}\sin^{2n}ax^2\mathrm{d}x=\int_0^{\infty}\cos^{2n}ax^2\mathrm{d}x=\infty$

293. $\int_0^{\infty}\sin^{2n+1}ax^2\mathrm{d}x=\frac{1}{2^{2n+1}}\sum_{k=0}^{n}(-1)^{n+k}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}\quad(a>0)$

294. $\int_0^{\infty}\cos^{2n+1}ax^2\mathrm{d}x=\frac{1}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}\quad(a>0)$

295. $\int_0^{\infty}\sin ax^2\sin 2bx\mathrm{d}x=\sqrt{\frac{\pi}{2a}}\left[\cos\frac{b^2}{a}\mathrm{C}\left(\frac{b}{\sqrt{a}}\right)+\sin\frac{b^2}{a}\mathrm{S}\left(\frac{b}{\sqrt{a}}\right)\right]\quad(a>0,\ b>0)$

[这里，S(z)和C(z)为菲涅耳函数(见附录)，以下同]

296. $\int_0^\infty \cos ax^2 \sin 2bx \mathrm{d}x = \sqrt{\frac{\pi}{2a}}\left[\sin\frac{b^2}{a}\mathrm{C}\left(\frac{b}{\sqrt{a}}\right) - \cos\frac{b^2}{a}\mathrm{S}\left(\frac{b}{\sqrt{a}}\right)\right]$ $(a>0,\ b>0)$

297. $\int_0^\infty \sin ax^2 \cos 2bx \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(\frac{b^2}{a} + \frac{\pi}{4}\right)$ $(a>0,\ b>0)$

298. $\int_0^\infty \cos ax^2 \cos 2bx \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$ $(a>0,\ b>0)$

299. $\int_0^\infty \cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right)\cos ax \mathrm{d}x = \sqrt{\frac{\pi}{2}}\cos\left(\frac{a^2}{2} - \frac{\pi}{8}\right)$ $(a>0)$

300. $\int_0^\infty \sin\frac{a^2}{x^2}\sin b^2x^2 \mathrm{d}x = \frac{1}{4b}\sqrt{\frac{\pi}{2}}(\sin 2ab - \cos 2ab + \mathrm{e}^{-2ab})$ $(a>0,\ b>0)$

301. $\int_0^\infty \cos\frac{a^2}{x^2}\sin b^2x^2 \mathrm{d}x = \frac{1}{4b}\sqrt{\frac{\pi}{2}}(\sin 2ab + \cos 2ab + \mathrm{e}^{-2ab})$ $(a>0,\ b>0)$

302. $\int_0^\infty \sin\frac{a^2}{x^2}\cos b^2x^2 \mathrm{d}x = \frac{1}{4b}\sqrt{\frac{\pi}{2}}(\sin 2ab + \cos 2ab + \mathrm{e}^{-2ab})$ $(a>0,\ b>0)$

303. $\int_0^\infty \cos\frac{a^2}{x^2}\cos b^2x^2 \mathrm{d}x = \frac{1}{4b}\sqrt{\frac{\pi}{2}}(\cos 2ab - \sin 2ab + \mathrm{e}^{-2ab})$ $(a>0,\ b>0)$

304. $\int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4a}(\cos 2ab + \sin 2ab)$ $(a>0,\ b>0)$

305. $\int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4a}(\cos 2ab - \sin 2ab)$ $(a>0,\ b>0)$

306. $\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4a}\mathrm{e}^{-2ab}$ $(a>0,\ b>0)$

307. $\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4a}\mathrm{e}^{-2ab}$ $(a>0,\ b>0)$

308. $\int_0^\infty \sin\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right)\mathrm{d}x = \int_0^\infty \cos\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4a}$ $(a>0,\ b>0)$

309. $\int_0^\infty (\sin^2 ax^2 - \sin^2 bx^2)\mathrm{d}x = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}}\right)$ $(a>0,\ b>0)$

310. $\int_0^\infty (\cos^2 ax^2 - \sin^2 bx^2)\mathrm{d}x = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{a}}\right)$ $(a>0,\ b>0)$

311. $\int_0^\infty (\cos^2 ax^2 - \cos^2 bx^2)\mathrm{d}x = \frac{1}{8}\left(\sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}}\right)$ $(a>0,\ b>0)$

312. $\int_0^\infty [\sin(a - x^2) + \cos(a - x^2)]\mathrm{d}x = \sqrt{\frac{\pi}{2}}\sin a$

313. $\int_0^{\frac{\pi}{2}} \sin(a\tan x)\mathrm{d}x = \frac{1}{2}[\mathrm{e}^{-a}\overline{\mathrm{Ei}}(a) - \mathrm{e}^{a}\mathrm{Ei}(-a)] \quad (a>0)$

{这里，$\mathrm{Ei}(x)$ 为指数积分（见附录），$\overline{\mathrm{Ei}}(x) = \frac{1}{2}[\mathrm{Ei}(x+\mathrm{i}0) + \mathrm{Ei}(x-\mathrm{i}0)]$ $(x>0)$，以下同}

314. $\int_0^{\frac{\pi}{2}} \cos(a\tan x)\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-a} \quad (a \geqslant 0)$

315. $\int_0^{\frac{\pi}{2}} \sin(a\tan x)\sin 2x\mathrm{d}x = \frac{a\pi}{2}\mathrm{e}^{-a} \quad (a \geqslant 0)$

316. $\int_0^{\frac{\pi}{2}} \cos(a\tan x)\sin^2 x\mathrm{d}x = \frac{(1-a)\pi}{4}\mathrm{e}^{-a} \quad (a \geqslant 0)$

317. $\int_0^{\frac{\pi}{2}} \cos(a\tan x)\cos^2 x\mathrm{d}x = \frac{(1+a)\pi}{4}\mathrm{e}^{-a} \quad (a \geqslant 0)$

318. $\int_0^{\frac{\pi}{2}} \sin(a\tan x)\tan x\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-a} \quad (a>0)$

319. $\int_0^{\frac{\pi}{2}} \cos(a\tan x)\tan x\mathrm{d}x = -\frac{1}{2}[\mathrm{e}^{-a}\overline{\mathrm{Ei}}(a) + \mathrm{e}^{a}\mathrm{Ei}(-a)] \quad (a>0)$

320. $\int_0^{\frac{\pi}{2}} \sin^2(a\tan x)\mathrm{d}x = \frac{\pi}{4}(1-\mathrm{e}^{-2a}) \quad (a \geqslant 0)$

321. $\int_0^{\frac{\pi}{2}} \cos^2(a\tan x)\mathrm{d}x = \frac{\pi}{4}(1+\mathrm{e}^{-2a}) \quad (a \geqslant 0)$

Ⅱ.1.2.14 三角函数与有理函数组合的积分

322. $\int_0^{\infty} \frac{\sin ax}{x+b}\mathrm{d}x = \sin ab\,\mathrm{ci}(ab) - \cos ab\,\mathrm{si}(ab) \quad (a>0,\ |\arg b| < \pi)$

[这里，$\mathrm{si}(x)$ 和 $\mathrm{ci}(x)$ 分别为正弦积分和余弦积分（见附录），以下同]

323. $\int_{-\infty}^{\infty} \frac{\sin ax}{x+b}\mathrm{d}x = \begin{cases} \pi\cos ab & (a>0) \\ \pi\mathrm{e}^{\mathrm{i}ab} & (a>0,\ \mathrm{Im}\ b>0) \\ \pi\mathrm{e}^{-\mathrm{i}ab} & (a>0,\ \mathrm{Im}\ b<0) \end{cases}$

324. $\int_0^{\infty} \frac{\cos ax}{x+b}\mathrm{d}x = -\sin ab\,\mathrm{si}(ab) - \cos ab\,\mathrm{ci}(ab) \quad (a>0,\ |\arg b| < \pi)$

325. $\int_{-\infty}^{\infty} \frac{\cos ax}{x+b}\mathrm{d}x = \begin{cases} \pi\sin ab & (a>0) \\ -\mathrm{i}\pi\mathrm{e}^{\mathrm{i}ab} & (a>0,\ \mathrm{Im}\ b>0) \\ \mathrm{i}\pi\mathrm{e}^{-\mathrm{i}ab} & (a>0,\ \mathrm{Im}\ b<0) \end{cases}$

326. $\int_0^{\infty} \frac{\sin ax}{b-x}\mathrm{d}x = \sin ab\,\mathrm{ci}(ab) - \cos ab[\mathrm{si}(ab) + \pi] \quad (a>0,\ b<0 \text{ 或 } \mathrm{Im}\ b \neq 0)$

327. $\int_{-\infty}^{\infty} \frac{\sin ax}{b-x}\mathrm{d}x = \begin{cases} -\pi\cos ab & (a>0) \\ -\pi\mathrm{e}^{\mathrm{i}ab} & (a>0,\ \mathrm{Im}\ b>0) \\ -\pi\mathrm{e}^{-\mathrm{i}ab} & (a>0,\ \mathrm{Im}\ b<0) \end{cases}$

328. $\int_0^\infty \frac{\cos ax}{b-x}\mathrm{d}x = \sin ab[\mathrm{si}(ab)+\pi] - \cos ab\,\mathrm{ci}(ab) \quad (a>0,\ b<0 \text{或} \operatorname{Im} b \neq 0)$

329. $\int_{-\infty}^\infty \frac{\cos ax}{b-x}\mathrm{d}x = \begin{cases} \pi\sin ab & (a>0) \\ -\mathrm{i}\pi\mathrm{e}^{\mathrm{i}ab} & (a>0,\ \operatorname{Im} b>0) \\ \mathrm{i}\pi\mathrm{e}^{-\mathrm{i}ab} & (a>0,\ \operatorname{Im} b<0) \end{cases}$

330. $\int_0^\infty \frac{\sin ax}{b^2+x^2}\mathrm{d}x = \frac{1}{2b}[\mathrm{e}^{-ab}\overline{\mathrm{Ei}}(ab) - \mathrm{e}^{ab}\mathrm{Ei}(-ab)] \quad (a>0,\ b>0)$

$\left\{\text{这里,}\mathrm{Ei}(x)\text{为指数积分(见附录),}\overline{\mathrm{Ei}}(x) = \frac{1}{2}[\mathrm{Ei}(x+\mathrm{i}0)+\mathrm{Ei}(x-\mathrm{i}0)]\right.$

$\left.(x>0)\text{,以下同}\right\}$

331. $\int_0^\infty \frac{\cos ax}{b^2+x^2}\mathrm{d}x = \frac{\pi}{2b}\mathrm{e}^{-ab} \quad (a\geqslant 0,\ \operatorname{Re} b>0)$

332. $\int_0^\infty \frac{\sin ax}{b^2-x^2}\mathrm{d}x = \frac{1}{b}\left\{\sin ab\,\mathrm{ci}(ab) - \cos ab\left[\mathrm{si}(ab)+\frac{\pi}{2}\right]\right\}$

$(a>0,\ |\arg b|<\pi)$

333. $\int_0^\infty \frac{\cos ax}{b^2-x^2}\mathrm{d}x = \frac{\pi}{2b}\sin ab \quad (a>0,\ b\neq 0)$

334. $\int_0^\infty \frac{\sin^2 ax}{b^2+x^2}\mathrm{d}x = \frac{\pi}{4b}(1-\mathrm{e}^{-2ab}) \quad (b\neq 0)$

335. $\int_0^\infty \frac{\cos^2 ax}{b^2+x^2}\mathrm{d}x = \frac{\pi}{4b}(1+\mathrm{e}^{-2ab}) \quad (b\neq 0)$

336. $\int_0^\infty \frac{\sin^2 ax}{b^2-x^2}\mathrm{d}x = -\frac{\pi}{4b}\sin 2ab \quad (a>0,\ b\neq 0)$

337. $\int_0^\infty \frac{\cos^2 ax}{b^2-x^2}\mathrm{d}x = \frac{\pi}{4b}\sin 2ab \quad (a>0,\ b>0)$

338. $\int_0^\infty \frac{x\sin ax}{b^2+x^2}\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-ab} \quad (a>0,\ \operatorname{Re} b>0)$

339. $\int_0^\infty \frac{x\cos ax}{b^2+x^2}\mathrm{d}x = -\frac{1}{2}[\mathrm{e}^{-ab}\overline{\mathrm{Ei}}(ab) + \mathrm{e}^{ab}\mathrm{Ei}(-ab)] \quad (a>0,\ b>0)$

340. $\int_0^\infty \frac{x\sin ax}{b^2-x^2}\mathrm{d}x = -\frac{\pi}{2}\cos ab \quad (a>0)$

341. $\int_0^\infty \frac{x\cos ax}{b^2-x^2}\mathrm{d}x = \cos ab\,\mathrm{ci}(ab) + \sin ab\left[\mathrm{si}(ab)+\frac{\pi}{2}\right] \quad (a>0,\ |\arg b|<\pi)$

342. $\int_0^\infty \frac{\sin ax\sin bx}{c^2+x^2}\mathrm{d}x = \frac{\pi}{2c}\mathrm{e}^{-ac}\sinh bc \quad (a\geqslant b)$

343. $\int_0^\infty \frac{\cos ax\cos bx}{c^2+x^2}\mathrm{d}x = \frac{\pi}{2c}\mathrm{e}^{-ac}\cosh bc \quad (a\geqslant b)$

344. $\int_{-\infty}^\infty \frac{\sin ax}{x(x-b)}\mathrm{d}x = \pi\frac{\cos ab-1}{b} \quad (a>0,\ b\neq 0)$

345. $\int_0^\infty \frac{\sin ax}{x(b^2+x^2)}\mathrm{d}x = \frac{\pi}{2b^2}(1-\mathrm{e}^{-ab}) \quad (a>0,\ \mathrm{Re}\ b>0)$

346. $\int_0^\infty \frac{\sin ax}{x(b^2-x^2)}\mathrm{d}x = \frac{\pi}{2b^2}(1-\cos ab) \quad (a>0)$

347. $\int_0^\infty \frac{\sin ax\cos bx}{x(x^2+p^2)}\mathrm{d}x = \begin{cases} \frac{\pi}{2p^2}\mathrm{e}^{-pb}\sinh pa & (b>a>0) \\ \frac{\pi}{2p^2}(1-\mathrm{e}^{-pa}\cosh pb) & (a>b>0) \end{cases}$

348. $\int_0^\infty \frac{\cos ax}{(b^2+x^2)^2}\mathrm{d}x = \frac{\pi}{4b^3}(1+ab)\mathrm{e}^{-ab} \quad (a>0,\ b>0)$

349. $\int_0^\infty \frac{x\sin ax}{(b^2+x^2)^2}\mathrm{d}x = \frac{\pi a}{4b}\mathrm{e}^{-ab} \quad (a>0,\ b>0)$

350. $\int_0^\infty \frac{1-x^2}{(1+x^2)^2}\cos px\mathrm{d}x = \frac{\pi p}{2}\mathrm{e}^{-p}$

351. $\int_0^\infty \frac{\sin ax\sin bx}{x}\mathrm{d}x = \frac{1}{4}\ln\left(\frac{a+b}{a-b}\right)^2 \quad (a>0,\ b>0,\ a\neq b)$

352. $\int_0^\infty \frac{\sin ax\cos bx}{x}\mathrm{d}x = \begin{cases} \frac{\pi}{2} & (a>b\geqslant 0) \\ \frac{\pi}{4} & (a=b>0) \\ 0 & (b>a\geqslant 0) \end{cases}$

353. $\int_0^\infty \frac{\sin ax\sin bx}{x^2}\mathrm{d}x = \begin{cases} \frac{a\pi}{2} & (0<a\leqslant b) \\ \frac{b\pi}{2} & (0<b\leqslant a) \end{cases}$

354. $\int_0^\infty \frac{\sin ax\sin bx}{p^2+x^2}\mathrm{d}x = \frac{\pi}{4p}[\mathrm{e}^{-|a-b|p}-\mathrm{e}^{-(a+b)p}] \quad (a>0,\ b>0,\ \mathrm{Re}\ p>0)$

355. $\int_0^\infty \frac{\cos ax\cos bx}{p^2+x^2}\mathrm{d}x = \frac{\pi}{4p}[\mathrm{e}^{-|a-b|p}+\mathrm{e}^{-(a+b)p}] \quad (a>0,\ b>0,\ \mathrm{Re}\ p>0)$

356. $\int_0^\infty \frac{\sin ax\sin bx}{p^2-x^2}\mathrm{d}x = \begin{cases} -\frac{\pi}{2p}\cos ap\sin bp & (a>b>0) \\ -\frac{\pi}{4p}\sin 2ap & (a=b>0) \\ -\frac{\pi}{2p}\sin ap\cos bp & (b>a>0) \end{cases}$

357. $\int_0^\infty \frac{\sin ax\cos bx}{p^2-x^2}x\mathrm{d}x = \begin{cases} -\frac{\pi}{2}\cos ap\cos bp & (a>b>0) \\ -\frac{\pi}{4}\cos 2ap & (a=b>0) \\ \frac{\pi}{2}\sin ap\sin bp & (b>a>0) \end{cases}$

358. $\int_0^\infty \frac{\cos ax\cos bx}{p^2-x^2}\mathrm{d}x = \begin{cases} \frac{\pi}{2p}\sin ap\cos bp & (a>b>0) \\ \frac{\pi}{4p}\sin 2ap & (a=b>0) \\ \frac{\pi}{2p}\cos ap\sin bp & (b>a>0) \end{cases}$

359. $\int_0^\infty \frac{\sin ax}{\sin bx}\cdot\frac{\mathrm{d}x}{x^2+p^2} = \frac{\pi}{2p}\cdot\frac{\sinh ap}{\sinh bp}$ $(b>a>0,\ \mathrm{Re}\ p>0)$

360. $\int_0^\infty \frac{\cos ax}{\cos bx}\cdot\frac{\mathrm{d}x}{x^2+p^2} = \frac{\pi}{2p}\cdot\frac{\cosh ap}{\cosh bp}$ $(b>a>0,\ \mathrm{Re}\ p>0)$

361. $\int_0^\infty \frac{\sin ax}{\cos bx}\cdot\frac{x\mathrm{d}x}{x^2+p^2} = -\frac{\pi}{2}\cdot\frac{\sinh ap}{\cosh bp}$ $(b>a>0,\ \mathrm{Re}\ p>0)$

362. $\int_0^\infty \frac{\cos ax}{\sin bx}\cdot\frac{x\mathrm{d}x}{x^2+p^2} = \frac{\pi}{2}\cdot\frac{\cosh ap}{\sinh bp}$ $(b>a>0,\ \mathrm{Re}\ p>0)$

363. $\int_0^\infty \frac{\sin ax}{\cos ax}\cdot\frac{x\mathrm{d}x}{x^2+b^2} = \int_0^\infty \frac{x\tan ax}{x^2+b^2}\mathrm{d}x = \frac{\pi}{\mathrm{e}^{2ab}+1}$ $(a>0,\ b>0)$

364. $\int_0^\infty \frac{\cos ax}{\sin ax}\cdot\frac{x\mathrm{d}x}{x^2+b^2} = \int_0^\infty \frac{x\cot ax}{x^2+b^2}\mathrm{d}x = \frac{\pi}{\mathrm{e}^{2ab}-1}$ $(a>0,\ b>0)$

365. $\int_0^\infty \frac{x}{\sin ax}\cdot\frac{\mathrm{d}x}{x^2+b^2} = \frac{\pi}{2\sin ab}$ $(b>0)$

Ⅱ.1.2.15 三角函数与无理函数组合的积分

366. $\int_0^\infty \frac{\sin ax}{\sqrt{x+p}}\mathrm{d}x = \sqrt{\frac{\pi}{2a}}[\cos ap - \sin ap + 2C(\sqrt{ap})\sin ap - 2S(\sqrt{ap})\cos ap]$
$(a>0,\ |\arg p|<\pi)$
[这里,S(x)和C(x)为菲涅耳积分(见附录),以下同]

367. $\int_0^\infty \frac{\cos ax}{\sqrt{x+p}}\mathrm{d}x = \sqrt{\frac{\pi}{2a}}[\cos ap + \sin ap - 2C(\sqrt{ap})\cos ap - 2S(\sqrt{ap})\sin ap]$
$(a>0,\ |\arg p|<\pi)$

368. $\int_0^\infty \frac{\sin ax}{\sqrt{x-u}}\mathrm{d}x = \sqrt{\frac{\pi}{2a}}(\sin au + \cos au)$ $(a>0,\ u>0)$

369. $\int_0^\infty \frac{\cos ax}{\sqrt{x-u}}\mathrm{d}x = \sqrt{\frac{\pi}{2a}}(\cos au - \sin au)$ $(a>0,\ u>0)$

370. $\int_0^\infty \frac{\sin ax}{\sqrt{x}}\mathrm{d}x = \int_0^\infty \frac{\cos ax}{\sqrt{x}}\mathrm{d}x = \sqrt{\frac{\pi}{2a}}$

371. $\int_0^\infty \frac{\sin ax}{\sqrt[p]{x}}\mathrm{d}x = \frac{\pi\sqrt[p]{a}}{2a\Gamma\left(\frac{1}{p}\right)\sin\frac{\pi}{2p}}$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

372. $\int_0^\infty \frac{\cos ax}{\sqrt[p]{x}}\mathrm{d}x = \frac{\pi\sqrt[p]{a}}{2a\Gamma\left(\frac{1}{p}\right)\cos\frac{\pi}{2p}}$

Ⅱ.1.2.16 三角函数与幂函数组合的积分

373. $\int_0^\infty x^{p-1}\sin ax\mathrm{d}x = \frac{\Gamma(p)}{a^p}\sin\frac{p\pi}{2} = \frac{\pi}{2a^p\Gamma(1-p)}\sec\frac{p\pi}{2}$

$(a>0,\ 0<\mathrm{Re}\ p<1)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

374. $\int_0^\infty x^{p-1}\cos ax\mathrm{d}x = \frac{\Gamma(p)}{a^p}\cos\frac{p\pi}{2} = \frac{\pi}{2a^p\Gamma(1-p)}\csc\frac{p\pi}{2}$

$(a>0,\ 0<\mathrm{Re}\ p<1)$

375. $\int_0^\infty x^p\sin(ax+b)\mathrm{d}x = \frac{\Gamma(1+p)}{a^{p+1}}\cos\left(b+\frac{p\pi}{2}\right)\quad(a>0,\ -1<p<0)$

376. $\int_0^\infty x^p\cos(ax+b)\mathrm{d}x = -\frac{\Gamma(1+p)}{a^{p+1}}\sin\left(b+\frac{p\pi}{2}\right)\quad(a>0,\ -1<p<0)$

377. $\int_0^\infty \frac{x^{p-1}}{q^2+x^2}\sin\left(ax-\frac{p\pi}{2}\right)\mathrm{d}x = -\frac{\pi}{2}q^{p-2}\mathrm{e}^{-aq}$

$(a>0,\ \mathrm{Re}\ q>0,\ 0<\mathrm{Re}\ p<2)$

378. $\int_0^\infty \frac{x^p}{q^2+x^2}\cos\left(ax-\frac{p\pi}{2}\right)\mathrm{d}x = \frac{\pi}{2}q^{p-1}\mathrm{e}^{-aq}$

$(a>0,\ \mathrm{Re}\ q>0,\ |\mathrm{Re}\ p|<1)$

379. $\int_0^\infty \frac{x^{p-1}}{x^2-b^2}\sin\left(ax-\frac{p\pi}{2}\right)\mathrm{d}x = \frac{\pi}{2}b^{p-2}\cos\left(ab-\frac{p\pi}{2}\right)$

$(a>0,\ b>0,\ 0<\mathrm{Re}\ p<2)$

380. $\int_0^\infty \frac{x^p}{x^2-b^2}\cos\left(ax-\frac{p\pi}{2}\right)\mathrm{d}x = -\frac{\pi}{2}b^{p-1}\sin\left(ab-\frac{p\pi}{2}\right)$

$(a>0,\ b>0,\ |p|<1)$

381. $\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}}\sin ax\mathrm{d}x = \frac{a}{b^p}\quad(a>0,\ b>0,\ p>0)$

382. $\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}}\cos ax\mathrm{d}x = \frac{p}{b^{p+1}}\quad(a>0,\ b>0,\ p>0)$

Ⅱ.1.2.17 三角函数的有理函数与 x 的有理函数组合的积分

383. $\int_0^\infty\left(\frac{\sin x}{x}-\frac{1}{1+x}\right)\frac{\mathrm{d}x}{x} = 1-\gamma$

[这里，γ 为欧拉常数(见附录)，以下同]

384. $\int_0^{\infty}\left(\cos x-\frac{1}{1+x}\right)\frac{\mathrm{d}x}{x}=-\gamma$

385. $\int_{-\infty}^{\infty}\frac{1-\cos ax}{x(x-b)}\mathrm{d}x=\frac{\pi\sin ab}{b}$ $(b>0,\ b\neq 0,\ b$ 为实数$)$

386. $\int_0^{\infty}\frac{\cos ax+x\sin ax}{1+x^2}\mathrm{d}x=\pi\mathrm{e}^{-a}$ $(a>0)$

387. $\int_0^{\frac{\pi}{2}}\left(\frac{1}{x}-\cot x\right)\mathrm{d}x=\ln\frac{\pi}{2}$

388. $\int_0^{\frac{\pi}{2}}\frac{x\sin x}{1-\cos x}\mathrm{d}x=\frac{\pi}{2}\ln 2+2G$

[这里，G 为卡塔兰常数(见附录)，以下同]

389. $\int_0^{\pi}\frac{x\sin x}{1-\cos x}\mathrm{d}x=2\pi\ln 2$

390. $\int_0^{\frac{\pi}{2}}\frac{\cos x\pm\sin x}{\cos x\mp\sin x}x\mathrm{d}x=\mp\frac{\pi}{4}\ln 2-G$

391. $\int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos x+\sin x}x\mathrm{d}x=\frac{\pi}{4}\ln 2-\frac{1}{2}G$

392. $\int_0^{\infty}\frac{\sin^{2m+1}x\sin 2mx}{a^2+x^2}\mathrm{d}x=\frac{(-1)^m\pi}{2^{2m+1}a}[(1-\mathrm{e}^{-2a})^{2m}-1]\sinh a$

$(a>0,\ m=1,2,\cdots)$

393. $\int_0^{\infty}\frac{\sin^{2m-1}x\sin(2m-1)x}{a^2+x^2}\mathrm{d}x=\frac{(-1)^{m+1}\pi}{2^{2m}a}(1-\mathrm{e}^{-2a})^{2m-1}$

$(a>0,\ m=1,2,\cdots)$

394. $\int_0^{\infty}\frac{\sin^{2m-1}x\sin(2m+1)x}{a^2+x^2}\mathrm{d}x=\frac{(-1)^{m-1}\pi}{2^{2m}a}\mathrm{e}^{-2a}(1-\mathrm{e}^{-2a})^{2m-1}$

$(a>0,\ m=1,2,\cdots)$

Ⅱ.1.2.18 三角函数的幂函数与 x 的幂函数组合的积分

395. $\int_0^{\pi}x\sin^p x\mathrm{d}x=\frac{\pi^2}{2^{p+1}}\frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2}$ $(p>-1)$

[这里，$\Gamma(z)$ 为伽马函数(见附录)]

396. $\int_0^{\pi}x\sin^{2m}x\mathrm{d}x=\int_0^{\pi}x\cos^{2m}x\mathrm{d}x=\frac{\pi^2}{2}\frac{(2m-1)!!}{(2m)!!}$

397. $\int_0^{\infty}\frac{\sin^{2n}ax-\sin^{2n}bx}{x}\mathrm{d}x=\frac{(2n-1)!!}{(2n)!!}\ln\frac{b}{a}$ $(ab>0,\ n=1,2,\cdots)$

398. $\int_0^{\infty}\frac{\cos^{2n}ax-\cos^{2n}bx}{x}\mathrm{d}x=\left[1-\frac{(2n-1)!!}{(2n)!!}\right]\ln\frac{b}{a}$ $(ab>0,\ n=1,2,\cdots)$

399. $\int_0^{\infty} \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x} \mathrm{d}x = \ln \frac{b}{a} \quad (ab > 0,\ m = 1,2,\cdots)$

400. $\int_0^{\infty} \frac{\cos^p ax \sin bx \cos x}{x} \mathrm{d}x = \frac{\pi}{2} \quad (b > ap,\ p > -1)$

401. $\int_0^{\infty} \frac{(3 - 4\sin^2 ax)\sin^2 ax}{x} \mathrm{d}x = \frac{1}{2}\ln 2 \quad (a \neq 0,\ a$ 为实数)

402. $\int_0^{\infty} \frac{\sin^{2m+1} x \cos^{2n} x}{x} \mathrm{d}x = \int_0^{\infty} \frac{\sin^{2m+1} x \cos^{2n-1} x}{x} \mathrm{d}x = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!}\pi$

$$= \frac{1}{2}\mathrm{B}\left(m + \frac{1}{2}, n + \frac{1}{2}\right)$$

[这里,$\mathrm{B}(p,q)$为贝塔函数(见附录)]

403. $\int_0^{\infty} \frac{\cos ax - \cos bx}{x} \mathrm{d}x = \ln \frac{b}{a} \quad (a > 0,\ b > 0)$

404. $\int_0^{\infty} \frac{(1 - \cos ax)\cos bx}{x} \mathrm{d}x = \ln \frac{\sqrt{|a^2 - b^2|}}{b} \quad (a > 0,\ b > 0,\ a \neq b)$

405. $\int_0^{\infty} \frac{\sin^2 ax - \sin^2 bx}{x} \mathrm{d}x = \frac{1}{2}\ln \frac{a}{b} \quad (a > 0,\ b > 0)$

406. $\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x = \frac{(b-a)\pi}{2} \quad (a \geqslant 0,\ b \geqslant 0)$

407. $\int_0^{\infty} \frac{a\sin bx - b\sin ax}{x^2} \mathrm{d}x = ab\ln \frac{a}{b} \quad (a > 0,\ b > 0)$

408. $\int_0^{\infty} \frac{\sin x - x\cos x}{x^2} \mathrm{d}x = 1$

409. $\int_0^{\infty} \frac{1 - \cos ax}{x^2} \mathrm{d}x = \frac{a\pi}{2} \quad (a \geqslant 0)$

410. $\int_0^{\infty} \frac{x^3 - \sin^3 x}{x^5} \mathrm{d}x = \frac{13}{32}\pi$

411. $\int_0^{\infty} \frac{(1 - \cos ax)\sin bx}{x^2} \mathrm{d}x = \frac{b}{2}\ln \frac{b^2 - a^2}{b^2} + \frac{a}{2}\ln \frac{a+b}{a-b} \quad (a > 0,\ b > 0)$

412. $\int_0^{\infty} \frac{(1 - \cos ax)\cos bx}{x^2} \mathrm{d}x = \begin{cases} \frac{\pi}{2}(a - b) & (0 < b \leqslant a) \\ 0 & (0 < a \leqslant b) \end{cases}$

413. $\int_0^{\frac{\pi}{2}} \frac{x}{1 + \sin x} \mathrm{d}x = \ln 2$

414. $\int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x} \mathrm{d}x = \frac{\pi}{2} - \ln 2$

415. $\int_0^{\frac{\pi}{2}} \frac{x^2}{1 - \cos x} \mathrm{d}x = -\frac{\pi^2}{4} + \pi\ln 2 + 4G = 3.3740473667\cdots$

[这里,G 为卡塔兰常数(见附录),以下同]

416. $\int_0^{\pi} \frac{x^2}{1 - \cos x} \mathrm{d}x = 4\pi\ln 2$

417. $\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin^2 x}\mathrm{d}x = \pi\ln 2$

418. $\int_0^{\frac{\pi}{4}} \frac{x^2}{\sin^2 x}\mathrm{d}x = -\frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 + G = 0.8435118417\cdots$

419. $\int_0^{\frac{\pi}{4}} \frac{x^2}{\cos^2 x}\mathrm{d}x = \frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 - G$

420. $\int_0^{\frac{\pi}{2}} \frac{x^2\cos x}{\sin^2 x}\mathrm{d}x = -\frac{\pi^2}{4} + 4G = 1.1964612764\cdots$

421. $\int_0^{\frac{\pi}{2}} \frac{x^3\cos x}{\sin^3 x}\mathrm{d}x = -\frac{\pi^2}{16} + \frac{3\pi}{2}\ln 2$

422. $\int_0^{\frac{\pi}{2}} \frac{x\cos^{p-1} x}{\sin^{p+1} x}\mathrm{d}x = \frac{\pi}{2p}\sec\frac{p\pi}{2} \quad (p < 1)$

423. $\int_0^{\frac{\pi}{4}} x\tan x\mathrm{d}x = -\frac{\pi}{8}\ln 2 + \frac{1}{2}G = 0.1857845358\cdots$

424. $\int_0^{\frac{\pi}{4}} x\cot x\mathrm{d}x = \frac{\pi}{8}\ln 2 + \frac{1}{2}G = 0.7301810584\cdots$

Ⅱ.1.2.19 含有 $\sin^n ax$，$\cos^n ax$，$\tan^n ax$ 和 $\frac{1}{x^m}$ 组合的积分，积分区间为$[0,\infty)$

这里，当没有特别说明时，m，n 为正整数；a，b，c，p，q 为正实数.

425. $\int_0^{\infty} \frac{\sin(\pm ax)}{x}\mathrm{d}x = \pm\frac{\pi}{2}$

426. $\int_0^{\infty} \frac{\tan(\pm ax)}{x}\mathrm{d}x = \pm\frac{\pi}{2}$

427. $\int_0^{\infty} \frac{\sin^2 ax}{x}\mathrm{d}x = \infty$

428. $\int_0^{\infty} \frac{\sin^3 ax}{x}\mathrm{d}x = \frac{\pi}{4}\operatorname{sgn} a$

429. $\int_0^{\infty} \frac{\sin^{2m} ax}{x}\mathrm{d}x = \int_0^{\infty} \frac{\cos^{2m} ax}{x}\mathrm{d}x = \infty$

430. $\int_0^{\infty} \frac{\sin^{2n+1} ax}{x}\mathrm{d}x = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2} \quad (n > 0)$

431. $\int_0^{\infty} \frac{\sin^2 ax}{x^2}\mathrm{d}x = \frac{a\pi}{2}$

432. $\int_0^{\infty} \frac{\sin^{2n} ax}{x^2}\mathrm{d}x = \frac{(2n-3)!!}{(2n-2)!!}\cdot\frac{a\pi}{2} \quad (n > 1)$

433. $\int_0^{\infty} \frac{\sin^3 ax}{x^2}\mathrm{d}x = \frac{3a}{4}\ln 3$

434. $\int_0^\infty \frac{\sin^3 ax}{x^3}\mathrm{d}x = \frac{3a^2\pi}{8}\operatorname{sgn} a$

435. $\int_0^\infty \frac{\sin^{2m+1} ax}{x^3}\mathrm{d}x = \frac{a^2\pi}{4}(2m+1)\frac{(2m-3)!!}{(2m)!!} \quad (a>0)$

436. $\int_0^\infty \frac{\sin^3 ax}{x^q}\mathrm{d}x = \frac{3-3^{q-1}}{4}a^{q-1}\cos\frac{q\pi}{2}\Gamma(1-q) \quad (a>0,\ 0<\operatorname{Re} q<2)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

437. $\int_0^\infty \frac{\sin^4 ax}{x^2}\mathrm{d}x = \frac{a\pi}{4}$

438. $\int_0^\infty \frac{\sin^4 ax}{x^3}\mathrm{d}x = a^2\ln 2$

439. $\int_0^\infty \frac{\sin^4 ax}{x^4}\mathrm{d}x = \frac{a^3\pi}{3} \quad (a>0)$

440. $\int_0^\infty \frac{\sin ax\sin bx}{x}\mathrm{d}x = \ln\sqrt{\frac{a+b}{a-b}} \quad (a\neq b)$

441. $\int_0^\infty \frac{\sin ax\cos bx}{x}\mathrm{d}x = \begin{cases} 0 & (b>a>0) \\ \frac{\pi}{2} & (a>b>0) \\ \frac{\pi}{4} & (a=b>0) \end{cases}$

442. $\int_0^\infty \frac{\cos ax\cos bx}{x}\mathrm{d}x = \infty$

443. $\int_0^\infty \frac{\sin ax\sin bx}{x^2}\mathrm{d}x = \begin{cases} \frac{a\pi}{2} & (0<a\leqslant b) \\ \frac{b\pi}{2} & (0<b\leqslant a) \end{cases}$

444. $\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$

445. $\int_0^\infty \frac{\cos x}{x}\mathrm{d}x = \infty$

446. $\int_0^\infty \frac{\tan x}{x}\mathrm{d}x = \frac{\pi}{2}$

447. $\int_0^\infty \frac{\sin x}{x^p}\mathrm{d}x = \frac{\pi}{2\Gamma(p)\sin\frac{p\pi}{2}} \quad (0<p<1)$

448. $\int_0^\infty \frac{\cos x}{x^p}\mathrm{d}x = \frac{\pi}{2\Gamma(p)\cos\frac{p\pi}{2}} \quad (0<p<1)$

449. $\int_0^\infty \frac{1-\cos px}{x^2}\mathrm{d}x = \frac{\pi|p|}{2}$

450. $\int_0^\infty \frac{\cos ax-\cos bx}{x}\mathrm{d}x = \ln\left|\frac{b}{a}\right|$

Ⅱ.1.2.20 含有$\sqrt{1 \pm k^2\sin^2 x}$和$\sqrt{1 \pm k^2\cos^2 x}$的积分

451. $\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2 x}\,\mathrm{d}x$

$$= \frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2 k^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^6}{5}-\cdots\right]$$

$$= \mathrm{E}(k) \quad (k^2<1)$$

[这里,E(k)为第二类完全椭圆积分(见附录),以下同]

452. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6+\cdots\right]$

$$= \mathrm{K}(k) \quad (k^2<1)$$

[这里,K(k)为第一类完全椭圆积分(见附录),以下同]

453. $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{(1-k^2\sin^2 x)^3}}$

$$= \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 3k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 5k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 7k^6+\cdots\right] \quad (k^2<1)$$

454. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{2k}\ln\frac{1+k}{1-k}$

455. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1-k^2\sin^2 x}}\mathrm{d}x = \frac{1}{k}\arcsin k$

456. $\int_0^{\infty} \sin x\sqrt{1-k^2\sin^2 x}\,\frac{\mathrm{d}x}{x} = \mathrm{E}(k)$

457. $\int_0^{\infty} \sin x\sqrt{1-k^2\cos^2 x}\,\frac{\mathrm{d}x}{x} = \mathrm{E}(k)$

458. $\int_0^{\infty} \tan x\sqrt{1-k^2\sin^2 x}\,\frac{\mathrm{d}x}{x} = \mathrm{E}(k)$

459. $\int_0^{\infty} \tan x\sqrt{1-k^2\cos^2 x}\,\frac{\mathrm{d}x}{x} = \mathrm{E}(k)$

460. $\int_0^{\infty} \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\frac{\mathrm{d}x}{x} = \int_0^{\infty} \frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{\mathrm{d}x}{x} = \mathrm{K}(k)$

461. $\int_0^{\infty} \frac{\tan x}{\sqrt{1-k^2\sin^2 x}}\frac{\mathrm{d}x}{x} = \int_0^{\infty} \frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\frac{\mathrm{d}x}{x} = \mathrm{K}(k)$

462. $\int_0^{\infty} \frac{\sin x}{\sqrt{1+\sin^2 x}}\frac{\mathrm{d}x}{x} = \int_0^{\infty} \frac{\sin x}{\sqrt{1+\cos^2 x}}\frac{\mathrm{d}x}{x} = \sqrt{\frac{1}{2}}\mathrm{K}\left(\sqrt{\frac{1}{2}}\right)$

463. $\int_0^{\infty} \frac{\tan x}{\sqrt{1+\sin^2 x}}\frac{\mathrm{d}x}{x} = \int_0^{\infty} \frac{\tan x}{\sqrt{1+\cos^2 x}}\frac{\mathrm{d}x}{x} = \frac{1}{\sqrt{2}}\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$

Ⅱ.1.2.21 更复杂自变数的三角函数与幂函数组合的积分

464. $\int_0^\infty x\sin ax^2\sin 2bx\mathrm{d}x = \frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a}+\sin\frac{b^2}{a}\right) \quad (a>0,\ b>0)$

465. $\int_0^\infty x\cos ax^2\sin 2bx\mathrm{d}x = \frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left(\sin\frac{b^2}{a}-\cos\frac{b^2}{a}\right) \quad (a>0,\ b>0)$

466. $\int_0^\infty \frac{\sin ax^2}{x^2}\mathrm{d}x = \sqrt{\frac{a\pi}{2}} \quad (a\geqslant 0)$

467. $\int_0^\infty \frac{\sin ax^2\cos bx^2}{x^2}\mathrm{d}x = \begin{cases} \frac{1}{2}\sqrt{\frac{\pi}{2}}(\sqrt{a+b}+\sqrt{a-b}) & (a>b>0) \\ \frac{1}{2}\sqrt{a\pi} & (a=b\geqslant 0) \\ \frac{1}{2}\sqrt{\frac{\pi}{2}}(\sqrt{a+b}-\sqrt{b-a}) & (b>a>0) \end{cases}$

468. $\int_0^1 \frac{\cos ax+\cos\frac{a}{x}}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{\cos ax+\cos\frac{a}{x}}{1+x^2}\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-a} \quad (a>0)$

469. $\int_0^1 \frac{\cos ax-\cos\frac{a}{x}}{1-x^2}\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{\cos ax-\cos\frac{a}{x}}{1-x^2}\mathrm{d}x = \frac{\pi}{2}\sin a \quad (a>0)$

470. $\int_0^\infty \sin\left(a^2x^2+\frac{b^2}{x^2}\right)\frac{\mathrm{d}x}{x^2} = \frac{\sqrt{\pi}}{2b}\sin\left(2ab+\frac{\pi}{4}\right) \quad (a>0,\ b>0)$

471. $\int_0^\infty \cos\left(a^2x^2+\frac{b^2}{x^2}\right)\frac{\mathrm{d}x}{x^2} = \frac{\sqrt{\pi}}{2b}\cos\left(2ab+\frac{\pi}{4}\right) \quad (a>0,\ b>0)$

472. $\int_0^\infty \sin\left(a^2x^2-\frac{b^2}{x^2}\right)\frac{\mathrm{d}x}{x^2} = -\frac{\sqrt{\pi}}{2b\sqrt{2}}\mathrm{e}^{-2ab} \quad (a>0,\ b>0)$

473. $\int_0^\infty \cos\left(a^2x^2-\frac{b^2}{x^2}\right)\frac{\mathrm{d}x}{x^2} = \frac{\sqrt{\pi}}{2b\sqrt{2}}\mathrm{e}^{-2ab} \quad (a>0,\ b>0)$

474. $\int_0^\infty \sin\left(ax-\frac{b}{x}\right)^2\frac{\mathrm{d}x}{x^2} = \frac{\sqrt{2\pi}}{4b} \quad (a>0,\ b>0)$

475. $\int_0^\infty \cos\left(ax-\frac{b}{x}\right)^2\frac{\mathrm{d}x}{x^2} = \frac{\sqrt{2\pi}}{4b} \quad (a>0,\ b>0)$

476. $\int_0^\infty \sin ax^p\frac{\mathrm{d}x}{x} = \frac{\pi}{2p} \quad (a>0,\ p>0)$

477. $\int_0^\infty \sin(a\tan x)\frac{\mathrm{d}x}{x} = \frac{\pi}{2}(1-\mathrm{e}^{-a}) \quad (a>0)$

478. $\int_0^1 x^{p-1}\sin(q\ln x)\mathrm{d}x = -\frac{q}{q^2+p^2} \quad (\mathrm{Re}\ p>|\ \mathrm{Im}\ q\ |)$

479. $\int_0^1 x^{p-1}\cos(q\ln x)\mathrm{d}x = \frac{q}{q^2+p^2}$ $(\mathrm{Re}\ p > |\ \mathrm{Im}\ q\ |)$

Ⅱ.1.2.22 三角函数与指数函数组合的积分

480. $\int_0^\infty \mathrm{e}^{-ax}\sin bx\mathrm{d}x = \frac{b}{a^2+b^2}$ $(a>0)$

481. $\int_0^\infty \mathrm{e}^{-ax}\cos bx\mathrm{d}x = \frac{a}{a^2+b^2}$ $(a>0)$

482. $\int_0^\infty \mathrm{e}^{-ax}\sin(bx+c)\mathrm{d}x = \frac{a\sin c+b\cos c}{a^2+b^2}$ $(a>0)$

483. $\int_0^\infty \mathrm{e}^{-ax}\cos(bx+c)\mathrm{d}x = \frac{a\cos c-b\sin c}{a^2+b^2}$ $(a>0)$

484. $\int_0^\infty \mathrm{e}^{-ax}\sin^2 bx\mathrm{d}x = \frac{2b^2}{a(a^2+4b^2)}$

485. $\int_0^\infty \mathrm{e}^{-ax}\cos^2 bx\mathrm{d}x = \frac{a^2+2b^2}{a(a^2+4b^2)}$

486. $\int_0^\infty \mathrm{e}^{-px^2}\sin bx\mathrm{d}x = \frac{b}{2p}\sum_{k=1}^{\infty}\frac{1}{(2k-1)!!}\left(-\frac{b^2}{2p}\right)^{k-1}$ $(\mathrm{Re}\ p>0)$

487. $\int_0^\infty \mathrm{e}^{-px^2}\cos bx\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{p}}\exp\left(-\frac{b^2}{4p}\right)$ $(\mathrm{Re}\ p>0)$

488. $\int_0^\infty \mathrm{e}^{-px^2}\sin ax\sin bx\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{p}}\left[\exp\left(-\frac{(a-b)^2}{4p}\right)-\exp\left(-\frac{(a+b)^2}{4p}\right)\right]$
$(\mathrm{Re}\ p>0)$

489. $\int_0^\infty \mathrm{e}^{-px^2}\cos ax\cos bx\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{p}}\left[\exp\left(-\frac{(a-b)^2}{4p}\right)+\exp\left(-\frac{(a+b)^2}{4p}\right)\right]$
$(\mathrm{Re}\ p>0)$

490. $\int_0^\infty \mathrm{e}^{-px^2}\sin ax^2\mathrm{d}x = \sqrt{\frac{\pi}{8}}\sqrt{\frac{\sqrt{p^2+a^2}-p}{p^2+a^2}} = \frac{\sqrt{\pi}}{2\sqrt[4]{p^2+a^2}}\sin\left(\frac{1}{2}\arctan\frac{a}{p}\right)$
$(\mathrm{Re}\ p>0,\ a>0)$

491. $\int_0^\infty \mathrm{e}^{-px^2}\cos ax^2\mathrm{d}x = \sqrt{\frac{\pi}{8}}\sqrt{\frac{\sqrt{p^2+a^2}+p}{p^2+a^2}} = \frac{\sqrt{\pi}}{2\sqrt[4]{p^2+a^2}}\cos\left(\frac{1}{2}\arctan\frac{a}{p}\right)$
$(\mathrm{Re}\ p>0,\ a>0)$

492. $\int_0^\infty \frac{\sin ax}{\mathrm{e}^{px}+1}\mathrm{d}x = \frac{1}{2a}-\frac{\pi}{2p\sinh\frac{a\pi}{p}}$ $(a>0,\ \mathrm{Re}\ p>0)$

493. $\int_0^\infty \frac{\sin ax}{\mathrm{e}^{px}-1}\mathrm{d}x = \frac{\pi}{2p}\coth\frac{a\pi}{p}-\frac{1}{2a}$ $(a>0,\ \mathrm{Re}\ p>0)$

494. $\int_0^{2\pi} e^{imx}\sin nx\,dx = \begin{cases} 0 & (m \neq n,\text{或 } m = n = 0) \\ i\pi & (m = n \neq 0) \end{cases}$

495. $\int_0^{2\pi} e^{imx}\cos nx\,dx = \begin{cases} 0 & (m \neq n) \\ \pi & (m = n \neq 0) \\ 2\pi & (m = n = 0) \end{cases}$

Ⅱ.1.2.23 三角函数与指数函数和幂函数组合的积分,积分区间为[0,∞)

496. $\int_0^{\infty} e^{-px}\sin qx\,\frac{dx}{x} = \arctan\frac{p}{q} \quad (p > 0)$

497. $\int_0^{\infty} e^{-px}\cos qx\,\frac{dx}{x} = \infty \quad (p > 0)$

498. $\int_0^{\infty} e^{-px}(1 - \cos ax)\,\frac{dx}{x} = \frac{1}{2}\ln\frac{a^2 + p^2}{p^2} \quad (\mathrm{Re}\ p > 0)$

499. $\int_0^{\infty} e^{-px}\sin qx\sin ax\,\frac{dx}{x} = \frac{1}{4}\ln\frac{p^2 + (a+q)^2}{p^2 + (a-q)^2} \quad (\mathrm{Re}\ p > |\mathrm{Im}\ q|,\ a > 0)$

500. $\int_0^{\infty} e^{-px}\sin ax\cos bx\,\frac{dx}{x} = \frac{1}{2}\arctan\frac{2pa}{p^2 - a^2 + b^2} + s\,\frac{\pi}{2} \quad (a \geqslant 0,\ p > 0)$

[这里,当 $p^2 - a^2 + b^2 \geqslant 0$ 时,$s = 0$;当 $p^2 - a^2 + b^2 < 0$ 时,$s = 1$]

501. $\int_0^{\infty} e^{-px}(\sin ax - \sin bx)\,\frac{dx}{x} = \arctan\frac{(a-b)p}{ab + p^2} \quad (\mathrm{Re}\ p > 0)$

502. $\int_0^{\infty} e^{-px}(\cos ax - \cos bx)\,\frac{dx}{x} = \frac{1}{2}\ln\frac{b^2 + p^2}{a^2 + p^2} \quad (\mathrm{Re}\ p > 0)$

503. $\int_0^{\infty} \left(e^{-qx} - e^{-px}\right)\sin bx\,\frac{dx}{x} = \arctan\frac{(p-q)b}{b^2 + pq} \quad (\mathrm{Re}\ p > 0,\ \mathrm{Re}\ q > 0)$

504. $\int_0^{\infty} \left(e^{-qx} - e^{-px}\right)\cos bx\,\frac{dx}{x} = \frac{1}{2}\ln\frac{b^2 + p^2}{b^2 + q^2} \quad (\mathrm{Re}\ p > 0,\ \mathrm{Re}\ q > 0)$

505. $\int_0^{\infty} (1 - e^{-x})\cos x\,\frac{dx}{x} = \ln\sqrt{2}$

506. $\int_0^{\infty} e^{-ax}\sin^2 bx\,\frac{dx}{x} = \ln\sqrt[4]{\frac{a^2 + 4b^2}{a^2}} \quad (a > 0)$

507. $\int_0^{\infty} e^{-ax}\sin^2 bx\,\frac{dx}{x^2} = b\arctan\frac{2b}{a} - a\ln\sqrt[4]{\frac{a^2 + 4b^2}{a^2}} \quad (a > 0)$

508. $\int_0^{\infty} e^{-ax}\cos^n bx\,\frac{dx}{x^m} = \infty \quad (a > 0)$

509. $\int_0^{\infty} \frac{x}{e^{px} - 1}\cos bx\,dx = \frac{1}{2b^2} - \frac{\pi^2}{2p^2}\cosh^2\frac{b\pi}{p} \quad (\mathrm{Re}\ p > 0)$

510. $\int_0^{\infty} x e^{-ax}\sin bx\,dx = \frac{2ab}{(a^2 + b^2)^2} \quad (a > 0)$

511. $\int_0^\infty x e^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} \quad (a>0)$

512. $\int_0^\infty x^n e^{-ax}\sin bx\,dx = \frac{n![(a+ib)^{n+1}-(a-ib)^{n+1}]}{2i(a^2+b^2)^{n+1}} \quad (a>0)$

513. $\int_0^\infty x^n e^{-ax}\cos bx\,dx = \frac{n![(a+ib)^{n+1}+(a-ib)^{n+1}]}{2i(a^2+b^2)^{n+1}} \quad (a>0,\ n>-1)$

Ⅱ.1.2.24 三角函数与三角函数的指数函数组合的积分

514. $\int_0^\pi e^{a\cos x}\sin x\,dx = \frac{2}{a}\sinh a$

515. $\int_0^\pi e^{p\cos x}\sin(p\sin x)\sin mx\,dx = \frac{\pi p^m}{2m!}$

516. $\int_0^\pi e^{p\cos x}\cos(p\sin x)\cos mx\,dx = \frac{\pi p^m}{2m!}$

517. $\int_0^{2\pi} e^{p\cos x}\cos(p\sin x - mx)\,dx = 2\int_0^\pi e^{p\cos x}\cos(p\sin x - mx)\,dx = \frac{2\pi p^m}{m!}$

518. $\int_0^{2\pi} e^{p\sin x}\sin(p\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\sin\frac{m\pi}{2} \quad (p>0)$

519. $\int_0^{2\pi} e^{p\sin x}\cos(p\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\cos\frac{m\pi}{2} \quad (p>0)$

520. $\int_0^{2\pi} e^{p\cos x}\sin(mx - \sin x)\,dx = 0$

521. $\int_0^\infty e^{-x\cos t}\cos(t - x\sin t)\,dx = 1$

522. $\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\frac{dx}{x} = \frac{\pi}{2}(e^p - 1) \quad (a>0,\ p>0)$

523. $\int_0^\infty \exp(p\cos ax)\sin(p\sin ax + nx)\frac{dx}{x} = \frac{\pi}{2}e^p \quad (p>0)$

Ⅱ.1.2.25 三角函数与双曲函数组合的积分

524. $\int_0^\infty \frac{\sin ax}{\sinh px}dx = \frac{\pi}{2p}\tanh\frac{a\pi}{2p} \quad (\mathrm{Re}\ p>0,\ a>0)$

525. $\int_0^\infty \frac{\cos ax}{\cosh px}dx = \frac{\pi}{2p}\mathrm{sech}\frac{a\pi}{2p}$ （Re $p>0$，a 可以是任何实数）

526. $\int_0^\infty \sin ax\frac{\sinh px}{\cosh qx}dx = \frac{\pi}{q}\cdot\frac{\sin\frac{p\pi}{2q}\sinh\frac{a\pi}{2q}}{\cos\frac{p\pi}{q}+\cosh\frac{a\pi}{q}} \quad (|\mathrm{Re}\ p|<\mathrm{Re}\ q,\ a>0)$

527. $\int_0^\infty \sin ax \frac{\cosh px}{\sinh qx} dx = \frac{\pi}{2q} \cdot \frac{\sinh \frac{a\pi}{q}}{\cos \frac{p\pi}{q} + \cosh \frac{a\pi}{q}}$ $(|\operatorname{Re} p| < \operatorname{Re} q,\ a > 0)$

528. $\int_0^\infty \cos ax \frac{\sinh px}{\sinh qx} dx = \frac{\pi}{2q} \cdot \frac{\sin \frac{p\pi}{q}}{\cos \frac{p\pi}{q} + \cosh \frac{a\pi}{q}}$ $(|\operatorname{Re} p| < \operatorname{Re} q)$

529. $\int_0^\infty \cos ax \frac{\cosh px}{\cosh qx} dx = \frac{\pi}{q} \cdot \frac{\cos \frac{p\pi}{2q} \cosh \frac{a\pi}{2q}}{\cos \frac{p\pi}{q} + \cosh \frac{a\pi}{q}}$

$(\operatorname{Re} p < \operatorname{Re} q$，$a$ 可以是任意实数$)$

Ⅱ.1.2.26 三角函数与双曲函数和幂函数组合的积分

530. $\int_0^\infty x \frac{\sin 2ax}{\cosh px} dx = \frac{\pi^2}{4p^2} \cdot \frac{\sinh \frac{a\pi}{p}}{\cosh^2 \frac{a\pi}{p}}$ $(\operatorname{Re} p > 0,\ a > 0)$

531. $\int_0^\infty x \frac{\cos 2ax}{\sinh px} dx = \frac{\pi^2}{4p^2} \cdot \frac{1}{\cosh^2 \frac{a\pi}{p}}$ $(\operatorname{Re} p > 0,\ a > 0)$

532. $\int_0^\infty \frac{\sin ax}{\cosh px} \cdot \frac{dx}{x} = 2\arctan\left[\exp\left(\frac{a\pi}{2p}\right)\right] - \frac{\pi}{2}$ $(\operatorname{Re} p > 0,\ a > 0)$

533. $\int_0^\infty \cos ax \tanh px \frac{dx}{x} = \ln\left(\coth \frac{a\pi}{4p}\right)$ $(\operatorname{Re} p > 0,\ a > 0)$

534. $\int_0^\infty \cos ax \coth px \frac{dx}{x} = -\ln\left(2\sinh \frac{a\pi}{4p}\right)$ $(\operatorname{Re} p > 0,\ a > 0)$

Ⅱ.1.2.27 三角函数与双曲函数、指数函数和幂函数组合的积分

535. $\int_0^\infty \sin ax \sinh px \exp\left(-\frac{x^2}{4q}\right) dx = \sqrt{q\pi} \sin(2apq) \exp[q(p^2 - a^2)]$ $(\operatorname{Re} q > 0)$

536. $\int_0^\infty \cos ax \cosh px \exp\left(-\frac{x^2}{4q}\right) dx = \sqrt{q\pi} \cos(2apq) \exp[q(p^2 - a^2)]$

$(\operatorname{Re} q > 0)$

537. $\int_0^\infty \frac{\cos ax \sinh px}{e^{qx} + 1} dx = -\frac{p}{2(a^2 + p^2)} + \frac{\pi}{q} \cdot \frac{\sin \frac{p\pi}{q} \cosh \frac{a\pi}{q}}{\cosh \frac{2a\pi}{q} - \cos \frac{2p\pi}{q}}$

$(|\operatorname{Re} p| < \operatorname{Re} q)$

538. $\int_0^\infty e^{-px^2}(\cosh x+\cos x)dx=\sqrt{\frac{\pi}{p}}\cosh\frac{1}{4p}\quad(\mathrm{Re}\ p>0)$

539. $\int_0^\infty e^{-px^2}(\cosh x-\cos x)dx=\sqrt{\frac{\pi}{p}}\sinh\frac{1}{4p}\quad(\mathrm{Re}\ p>0)$

540. $\int_0^\infty xe^{-px^2}\sinh x\cos x dx=\frac{1}{4}\sqrt{\frac{\pi}{p^3}}\left(\cos\frac{1}{2p}-\sin\frac{1}{2p}\right)\quad(\mathrm{Re}\ p>0)$

541. $\int_0^\infty xe^{-px^2}\cosh x\sin x dx=\frac{1}{4}\sqrt{\frac{\pi}{p^3}}\left(\cos\frac{1}{2p}+\sin\frac{1}{2p}\right)\quad(\mathrm{Re}\ p>0)$

542. $\int_0^\infty x^2e^{-px^2}\sinh x\sin x dx=\frac{1}{4}\sqrt{\frac{\pi}{p^3}}\left(\sin\frac{1}{2p}+\frac{1}{p}\cos\frac{1}{2p}\right)\quad(\mathrm{Re}\ p>0)$

543. $\int_0^\infty x^2e^{-px^2}\cosh x\cos x dx=\frac{1}{4}\sqrt{\frac{\pi}{p^3}}\left(\cos\frac{1}{2p}-\frac{1}{p}\sin\frac{1}{2p}\right)\quad(\mathrm{Re}\ p>0)$

Ⅱ.1.2.28 反三角函数与幂函数和代数函数组合的积分

544. $\int_0^1 x^{2n}\arcsin x dx=\frac{1}{2n+1}\left[\frac{\pi}{2}-\frac{2^n n!}{(2n+1)!!}\right]$

545. $\int_0^1 x^{2n}\arccos x dx=\frac{2^n n!}{(2n+1)(2n+1)!!}$

546. $\int_0^1 x^{2n-1}\arcsin x dx=\frac{\pi}{4n}\left[1-\frac{(2n-1)!!}{2^n n!}\right]$

547. $\int_0^1 x^{2n-1}\arccos x dx=\frac{\pi}{4n}\frac{(2n-1)!!}{2^n n!}$

548. $\int_{-1}^1(1-x^2)^n\arccos x dx=\pi\frac{2^n n!}{(2n+1)!!}$

549. $\int_0^1\left(\frac{\pi}{4}-\arctan x\right)\frac{dx}{1-x}=-\frac{\pi}{8}\ln 2+G$

[这里，G 为卡塔兰常数(见附录)，以下同]

550. $\int_0^\infty x^p\operatorname{arccot}x dx=-\frac{\pi}{2(p+1)}\csc\frac{p\pi}{2}\quad(-1<p<0)$

551. $\int_0^\infty(1-x\operatorname{arccot}x)dx=\frac{\pi}{4}$

552. $\int_0^1\frac{\arcsin x}{x}dx=\frac{\pi}{2}\ln 2$

553. $\int_0^1\frac{\arccos x}{x}dx=-\frac{\pi}{2}\ln 2$

554. $\int_0^1\frac{\arctan x}{x}dx=\int_1^\infty\frac{\operatorname{arccot}x}{x}dx=G$

555. $\int_0^1\frac{\arccos x}{1\pm x}dx=\mp\frac{\pi}{2}\ln 2+2G$

556. $\int_0^\infty \frac{\operatorname{arccot} x}{1 \pm x} dx = \pm \frac{\pi}{4} \ln 2 + G$

557. $\int_0^1 \arcsin x \frac{dx}{x(1+qx^2)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \quad (q > -1)$

558. $\int_0^1 \arcsin x \frac{x}{1+qx^2} dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \quad (q > -1)$

559. $\int_0^1 \arcsin x \frac{x}{1-p^2x^2} dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \quad (p^2 < 1)$

560. $\int_0^1 \arcsin x \frac{x}{(1+qx^2)^2} dx = \frac{\pi}{4q} \ln \frac{\sqrt{1+q}-1}{1+q} \quad (q > -1)$

561. $\int_0^1 \arccos x \frac{x}{(1+qx^2)^2} dx = \frac{\pi}{4q} \ln \frac{\sqrt{1+q}-1}{\sqrt{1+q}} \quad (q > -1)$

562. $\int_0^1 \arccos x \frac{dx}{\sin^2\lambda - x^2} = 2\csc\lambda \sum_{k=0}^{\infty} \frac{\sin[(2k-1)\lambda]}{(2k+1)^2}$

563. $\int_0^1 \frac{\arctan x}{x(1+x)} dx = -\frac{\pi}{8} \ln 2 + G$

564. $\int_0^\infty \frac{\arctan x}{1-x^2} dx = -G$

565. $\int_0^\infty \frac{x \arctan x}{1+x^4} dx = \frac{\pi^2}{16}$

566. $\int_0^\infty \frac{x \arctan x}{1-x^4} dx = -\frac{\pi}{8} \ln 2$

567. $\int_0^\infty \frac{x \operatorname{arccot} x}{1-x^4} dx = \frac{\pi}{8} \ln 2$

568. $\int_0^\infty \frac{\arctan x}{x\sqrt{1+x^2}} dx = \int_0^\infty \frac{\operatorname{arccot} x}{\sqrt{1+x^2}} dx = 2G$

569. $\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(1+\sqrt{2})$

570. $\int_0^1 \frac{(\arcsin x)^2}{x^2\sqrt{1-x^2}} dx = \pi \ln 2$

571. $\int_0^1 \frac{(\arccos x)^2}{(\sqrt{1-x^2})^3} dx = \pi \ln 2$

572. $\int_0^\infty \frac{(\arctan x)^2}{x^2\sqrt{1+x^2}} dx = \int_0^\infty \frac{x(\operatorname{arccot} x)^2}{\sqrt{1+x^2}} dx = -\frac{\pi^2}{4} + 4G$

573. $\int_0^1 \frac{\arctan qx}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(q + \sqrt{1+q^2})$

574. $\int_0^1 \frac{\arctan px}{1+p^2x} dx = \frac{1}{2p^2} \arctan p \ln(1+p^2)$

575. $\int_0^1 \frac{\operatorname{arccot} px}{1+p^2x}\mathrm{d}x = \frac{1}{p^2}\left(\frac{\pi}{4}+\frac{1}{2}\operatorname{arccot} p\right)\ln(1+p^2) \quad (p>0)$

576. $\int_0^\infty \frac{\arctan qx}{(p+x)^2}\mathrm{d}x = -\frac{q}{1+p^2q^2}\left(\ln pq - \frac{\pi}{2}pq\right) \quad (p>0,\ q>0)$

577. $\int_0^\infty \frac{\operatorname{arccot} qx}{(p+x)^2}\mathrm{d}x = \frac{q}{1+p^2q^2}\left(\ln pq + \frac{\pi}{2pq}\right) \quad (p>0,\ q>0)$

578. $\int_0^\infty \frac{\arctan px}{x(1+x^2)}\mathrm{d}x = \frac{\pi}{2}\ln(1+p) \quad (p\geqslant 0)$

579. $\int_0^\infty \frac{\arctan px}{x(1-x^2)}\mathrm{d}x = \frac{\pi}{4}\ln(1+p^2) \quad (p\geqslant 0)$

580. $\int_0^\infty \frac{x\operatorname{arccot} px}{q^2+x^2}\mathrm{d}x = \frac{\pi}{2}\ln\frac{1+pq}{pq} \quad (p>0,\ q>0)$

581. $\int_0^\infty \frac{x\operatorname{arccot} px}{x^2-p^2}\mathrm{d}x = \frac{\pi}{4}\ln\frac{1+p^2q^2}{p^2q^2} \quad (p>0,\ q>0)$

582. $\int_0^\infty \frac{x\arctan qx}{(p^2+x^2)^2}\mathrm{d}x = \frac{q\pi}{4p(1+pq)} \quad (p>0,\ q\geqslant 0)$

583. $\int_0^\infty \frac{x\operatorname{arccot} qx}{(p^2+x^2)^2}\mathrm{d}x = \frac{\pi}{4p^2(1+pq)} \quad (p>0,\ q\geqslant 0)$

584. $\int_0^\infty \frac{\arctan px - \arctan qx}{x}\mathrm{d}x = \frac{\pi}{2}\ln\frac{p}{q} \quad (p>0,\ q>0)$

585. $\int_0^\infty \frac{\arctan px\arctan qx}{x^2}\mathrm{d}x = \frac{\pi}{2}\ln\frac{(p+q)^{p+q}}{p^pq^q} \quad (p>0,\ q>0)$

586. $\int_0^\infty \frac{\arctan x^2}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\arctan x^3}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\operatorname{arccot} x^2}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\operatorname{arccot} x^3}{1+x^2}\mathrm{d}x = \frac{\pi^2}{8}$

Ⅱ.1.2.29 反三角函数与三角函数、指数函数和对数函数组合的积分

587. $\int_0^\infty \operatorname{arccot} qx\sin px\mathrm{d}x = \frac{\pi}{2p}\left(1-\mathrm{e}^{-\frac{p}{q}}\right) \quad (p>0,\ q>0)$

588. $\int_0^\infty \operatorname{arccot} qx\cos px\mathrm{d}x = \frac{1}{2p}\left[\mathrm{e}^{-\frac{p}{q}}\mathrm{Ei}\left(\frac{p}{q}\right) - \mathrm{e}^{\frac{p}{q}}\mathrm{Ei}\left(-\frac{p}{q}\right)\right] \quad (p>0,\ q>0)$

[这里，$\mathrm{Ei}(z)$为指数积分(见附录)，以下同]

589. $\int_0^\infty \arctan\frac{2a}{x}\sin bx\mathrm{d}x = \frac{\pi}{b}\mathrm{e}^{-ab}\sinh ab \quad (\mathrm{Re}\ a>0,\ b>0)$

590. $\int_0^\infty \arctan\frac{a}{x}\cos bx\mathrm{d}x = \frac{1}{2b}\left[\mathrm{e}^{-ab}\overline{\mathrm{Ei}}(ab) - \mathrm{e}^{ab}\mathrm{Ei}(-ab)\right] \quad (a>0,\ b>0)$

[这里，$\overline{\mathrm{Ei}}(z)$为指数积分(见附录)]

591. $\int_0^\infty \arctan\frac{2}{x^2}\cos bx\mathrm{d}x = \frac{\pi}{b}\mathrm{e}^{-b}\sin b \quad (b>0)$

592. $\int_0^\infty \left(\frac{2}{\pi}\operatorname{arccot} x - \cos px\right)\mathrm{d}x = \ln p + \gamma$

[这里，γ 为欧拉常数(见附录)]

593. $\int_0^\infty \frac{\operatorname{arccot} px \tan x}{q^2\cos^2 x + r^2\sin^2 x}\mathrm{d}x = \frac{\pi}{2r^2}\ln\left(1+\frac{r}{q}\tanh\frac{1}{p}\right)$ $(p>0,\ q>0,\ r>0)$

594. $\int_0^\infty \mathrm{e}^{-ax}\arctan\frac{x}{b}\mathrm{d}x = \frac{1}{a}[-\operatorname{ci}(ab)\sin ab - \operatorname{si}(ab)\cos ab]$ $(\operatorname{Re} a>0)$

[这里，$\operatorname{si}(z)$ 和 $\operatorname{ci}(z)$ 分别为正弦积分和余弦积分(见附录)，以下同]

595. $\int_0^\infty \mathrm{e}^{-ax}\operatorname{arccot}\frac{x}{b}\mathrm{d}x = \frac{1}{a}\left[\frac{\pi}{2}+\operatorname{ci}(ab)\sin ab + \operatorname{si}(ab)\cos ab\right]$ $(\operatorname{Re} a>0)$

596. $\int_0^1 \arcsin x\ln x\mathrm{d}x = 2-\ln 2-\frac{\pi}{2}$

597. $\int_0^1 \arccos x\ln x\mathrm{d}x = \ln 2-2$

598. $\int_0^1 \arctan x\ln x\mathrm{d}x = \frac{1}{2}\ln 2-\frac{\pi}{4}+\frac{\pi^2}{48}$

599. $\int_0^1 \operatorname{arccot} x\ln x\mathrm{d}x = -\frac{1}{2}\ln 2-\frac{\pi}{4}-\frac{\pi^2}{48}$

600. $\int_0^1 \frac{\arccos x}{\ln x}\mathrm{d}x = -\sum_{k=0}^{\infty}\frac{(2k-1)!!\ln(2k+2)}{2^k k!(2k+1)}$

Ⅱ.1.3 指数函数和对数函数的定积分

Ⅱ.1.3.1 含有 e^{ax}，e^{-ax}，e^{-ax^2} 的积分

601. $\int_0^\infty \mathrm{e}^{-ax}\mathrm{d}x = \frac{1}{a}$ $(a>0)$

602. $\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x = 1$

603. $\int_0^\infty x^{n-1}\mathrm{e}^{-x}\mathrm{d}x = \Gamma(n)$ (n 为正整数)

[这里，$\Gamma(z)$ 为伽马函数(见附录)，以下同]

604. $\int_0^\infty x^n p^{-x}\mathrm{d}x = \frac{n!}{(\ln p)^{n+1}}$ ($p>0$，n 为大于 0 的整数)

605. $\int_0^\infty x^{n-1}\mathrm{e}^{-(a+1)x}\mathrm{d}x = \frac{\Gamma(n)}{(a+1)^n}$ $(n>0,\ a>-1)$

606. $\int_0^\infty x^n\mathrm{e}^{-ax}\mathrm{d}x = \frac{n!}{a^{n+1}}$ $(a>0)$

607. $\int_0^1 x^n\mathrm{e}^{-ax}\mathrm{d}x = \frac{n!}{a^{n+1}}\left(1-\mathrm{e}^{-a}\sum_{k=0}^{n}\frac{a^k}{k!}\right)$ $(a>0)$

608. $\int_0^1 x^{-x}\mathrm{d}x = \int_0^1 \mathrm{e}^{-x\ln x}\mathrm{d}x = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627\cdots$

609. $\int_0^{\infty} \mathrm{e}^{-a^2x^2}\mathrm{d}x = \frac{1}{2a}\sqrt{\pi} \quad (a>0)$

610. $\int_0^b \mathrm{e}^{-ax^2}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{erf}(b\sqrt{a}) \quad (a>0)$

[这里,erf(x)为误差函数(见附录)]

611. $\int_b^{\infty} \mathrm{e}^{-ax^2}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{erfc}(b\sqrt{a}) \quad (a>0)$

[这里,erfc(x)为补余误差函数(见附录)]

612. $\int_0^{\infty} \mathrm{e}^{-ax^2}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a>0)$

613. $\int_0^{\infty} x\mathrm{e}^{-ax^2}\mathrm{d}x = \frac{1}{2a} \quad (a>0)$

614. $\int_0^{\infty} x^2\mathrm{e}^{-ax^2}\mathrm{d}x = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \quad (a>0)$

615. $\int_0^{\infty} x^n\mathrm{e}^{-ax^2}\mathrm{d}x = \frac{(n-1)!!}{2(2a)^{\frac{n}{2}}}\sqrt{\frac{\pi}{a}} \quad (a>0,\ n>0)$

616. $\int_0^{\infty} x^n\mathrm{e}^{-ax^p}\mathrm{d}x = \frac{\Gamma\left(\frac{n+1}{p}\right)}{pa^{\frac{n+1}{p}}} \quad (a>0,\ p>0,\ n>-1)$

617. $\int_0^{\infty} \frac{\mathrm{e}^{-ax}}{x}\mathrm{d}x = \infty \quad (a>0)$

618. $\int_0^{\infty} \frac{\mathrm{d}x}{\mathrm{e}^{ax}+1} = \frac{\ln 2}{a} \quad (a>0)$

619. $\int_0^{\infty} \frac{\mathrm{d}x}{\mathrm{e}^{ax}-1} = \infty \quad (a>0)$

620. $\int_0^{\infty} \frac{x}{\mathrm{e}^{ax}+1}\mathrm{d}x = \frac{\pi^2}{12a^2} \quad (a>0)$

621. $\int_0^{\infty} \frac{x}{\mathrm{e}^{ax}-1}\mathrm{d}x = \frac{\pi^2}{6a^2} \quad (a>0)$

622. $\int_0^{\infty} \frac{x^2}{\mathrm{e}^{ax}+1}\mathrm{d}x = \frac{3}{2a^3}\sum_{k=1}^{\infty}\frac{1}{k^3} \quad (a>0)$

623. $\int_0^{\infty} \frac{x^2}{\mathrm{e}^{ax}-1}\mathrm{d}x = \frac{2}{a^3}\sum_{k=1}^{\infty}\frac{1}{k^3} \quad (a>0)$

624. $\int_0^{\infty} \frac{x^p}{\mathrm{e}^{ax}+1}\mathrm{d}x = \frac{\Gamma(p+1)}{a^{p+1}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{p+1}} \quad (a>0)$

625. $\int_0^{\infty} \frac{x^p}{\mathrm{e}^{ax}-1}\mathrm{d}x = \frac{\Gamma(p+1)}{a^{p+1}}\sum_{k=1}^{\infty}\frac{1}{k^{p+1}} \quad (a>0)$

626. $\int_{-\infty}^{\infty} \frac{e^{-px}}{1+e^{-qx}}dx = \frac{\pi}{q}\csc\frac{p\pi}{q}$ （$q>p>0$，或 $q<p<0$）

627. $\int_{0}^{\infty} \frac{e^{-px}}{1-ae^{-qx}}dx = \sum_{k=0}^{\infty}\frac{a^k}{p+kq}$ （$0<a<1$）

628. $\int_{-\infty}^{\infty} \frac{e^{-px}}{b+e^{-x}}dx = \pi b^{p-1}\csc p\pi$ （$|\arg b|<\pi$，$0<\mathrm{Re}\ p<1$）

629. $\int_{-\infty}^{\infty} \frac{e^{-px}}{b-e^{-x}}dx = \pi b^{p-1}\cot p\pi$ （$b>0$，$0<\mathrm{Re}\ p<1$）

630. $\int_{0}^{\infty} \frac{e^{-ax}-e^{-bx}}{x}dx = \ln\frac{b}{a}$ （$a>0$，$b>0$）

631. $\int_{0}^{\infty} \frac{e^{-ax^2}-e^{-bx^2}}{x}dx = \ln\sqrt{\frac{b}{a}}$ （$a>0$，$b>0$）

632. $\int_{0}^{\infty} \frac{dx}{e^{nx}+e^{-nx}} = \frac{\pi}{4n}$

633. $\int_{0}^{\infty} \frac{x}{e^{nx}-e^{-nx}}dx = \frac{\pi^2}{8n^2}$

Ⅱ.1.3.2 含有更复杂自变数的指数函数的积分

634. $\int_{-\infty}^{\infty} \exp(-p^2x^2 \pm qx)dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p}$ （$p>0$）

635. $\int_{0}^{\infty} \exp\left(-x^2-\frac{a^2}{x^2}\right)dx = \frac{e^{-2|a|}\sqrt{\pi}}{2}$

636. $\int_{0}^{\infty} \exp\left(-ax^2-\frac{b}{x^2}\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp(-2\sqrt{ab})$ （$a>0$，$b>0$）

637. $\int_{-\infty}^{\infty} \exp(-e^x)e^{px}dx = \Gamma(p)$ （$\mathrm{Re}\ p>0$）

［这里，$\Gamma(z)$为伽马函数（见附录）］

638. $\int_{0}^{\pi} \exp(z\cos x)dx = \pi I_0(z)$

［这里，$I_\nu(z)$为第一类修正贝塞尔函数（见附录）］

639. $\int_{0}^{\frac{\pi}{2}} \exp(-q\sin x)\sin 2x dx = \frac{2}{q^2}[(q-1)e^q+1]$

Ⅱ.1.3.3 指数函数的有理式与幂函数和有理函数组合的积分

640. $\int_{0}^{\infty} \frac{xe^{-x}}{e^x-1}dx = \frac{\pi^2}{6}-1$

641. $\int_{0}^{\infty} \frac{xe^{-2x}}{e^{-x}+1}dx = 1-\frac{\pi^2}{12}$

642. $\int_0^\infty \frac{xe^{-2nx}}{1+e^x}dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k^2}$

643. $\int_0^\infty \frac{xe^{-(2n-1)x}}{1+e^x}dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^k}{k^2}$

644. $\int_0^\infty \frac{x^2e^{-nx}}{1+e^{-x}}dx = 2\sum_{k=n}^{\infty}\frac{(-1)^{n+k}}{k^3} \quad (n=1,2,\cdots)$

645. $\int_0^\infty \frac{x^2e^{-nx}}{1-e^{-x}}dx = 2\sum_{k=n}^{\infty}\frac{1}{k^3} \quad (n=1,2,\cdots)$

646. $\int_0^\infty \frac{x^3e^{-nx}}{1+e^{-x}}dx = 6\sum_{k=n}^{\infty}\frac{(-1)^{n+k}}{k^4}$

647. $\int_0^\infty \frac{x^3e^{-nx}}{1-e^{-x}}dx = \frac{\pi^4}{15} - 6\sum_{k=1}^{n-1}\frac{1}{k^4}$

648. $\int_0^\infty \frac{x^{n-1}(1-e^{-mx})}{1-e^x}dx = (n-1)!\sum_{k=1}^{m}\frac{1}{k^n}$

649. $\int_0^\infty \frac{x^{p-1}}{e^{ax}-q}dx = \frac{1}{qa^p}\Gamma(p)\sum_{k=1}^{\infty}\frac{q^k}{k^p} \quad (p>0,\ a>0,\ -1<q<1)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

650. $\int_{-\infty}^\infty \frac{xe^{px}}{b+e^x}dx = \pi b^{p-1}\csc p\pi(\ln b - \pi\cot p\pi)$

$(|\arg b|<\pi,\ 0<\mathrm{Re}\ p<1)$

651. $\int_{-\infty}^\infty \frac{xe^{px}}{e^{qx}-1}dx = \frac{\pi^2}{q^2}\csc^2\frac{p\pi}{q} \quad (\mathrm{Re}\ q>\mathrm{Re}\ p>0)$

652. $\int_0^\infty x\frac{1+e^{-x}}{e^x-1}dx = \frac{\pi^2}{3}-1$

653. $\int_0^\infty x\frac{1-e^{-x}}{1+e^{-3x}}e^{-x}dx = \frac{2\pi^2}{27}$

654. $\int_0^\infty \frac{1-e^{-px}}{1+e^x}\frac{dx}{x} = \ln\frac{\sqrt{\pi}\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p+1}{2}\right)} \quad (\mathrm{Re}\ p>-1)$

655. $\int_0^\infty \frac{e^{-qx}-e^{-px}}{1+e^{-x}}\frac{dx}{x} = \ln\frac{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q+1}{2}\right)} \quad (\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

656. $\int_{-\infty}^\infty \frac{e^{px}-e^{qx}}{1+e^{rx}}\frac{dx}{x} = \ln\left(\tan\frac{p\pi}{2r}\cot\frac{q\pi}{2r}\right)$

$(|r|>|p|,\ |r|>|q|,\ rp>0,\ rq>0)$

657. $\int_{-\infty}^\infty \frac{e^{px}-e^{qx}}{1-e^{rx}}\frac{dx}{x} = \ln\left(\sin\frac{p\pi}{r}\csc\frac{q\pi}{r}\right)$

$(|r|>|p|,\ |r|>|q|,\ rp>0,\ rq>0)$

658. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x^{p+1}}dx = \frac{b^p-a^p}{p}\Gamma(1-p)$ $(\mathrm{Re}\, a>0,\ \mathrm{Re}\, b<0,\ \mathrm{Re}\, p<1)$

659. $\int_{-\infty}^\infty \frac{x}{a^2e^x+b^2e^{-x}}dx = \frac{\pi}{2ab}\ln\frac{b}{a}$ $(ab>0)$

660. $\int_{-\infty}^\infty \frac{x}{a^2e^x-b^2e^{-x}}dx = \frac{\pi^2}{4ab}$

661. $\int_0^\infty \frac{x}{e^x+e^{-x}-1}dx = \frac{1}{3}\left[\psi'\left(\frac{1}{3}\right)-\frac{2}{3}\pi^2\right] = 1.1719536193\cdots$

[这里，$\psi'(z)$ 为 ψ 函数的微商(见附录)，以下同]

662. $\int_0^\infty \frac{xe^{-x}}{e^x+e^{-x}-1}dx = \frac{1}{6}\left[\psi'\left(\frac{1}{3}\right)-\frac{5}{6}\pi^2\right] = 0.3118211319\cdots$

663. $\int_0^{\ln 2} \frac{x}{e^x+2e^{-x}-2}dx = \frac{\pi}{8}\ln 2$

664. $\int_{-\infty}^\infty \frac{x}{(a+e^x)(1+e^{-x})}dx = \frac{(\ln a)^2}{2(a-1)}$ $(|\arg a|<\pi)$

665. $\int_{-\infty}^\infty \frac{x}{(a+e^x)(1-e^{-x})}dx = \frac{\pi^2+(\ln a)^2}{2(a+1)}$ $(|\arg a|<\pi)$

666. $\int_{-\infty}^\infty \frac{xe^x}{(a+e^x)^2}dx = \frac{1}{a}\ln a$ $(|\arg a|<\pi)$

667. $\int_{-\infty}^\infty \frac{a^2e^x+b^2e^{-x}}{(a^2e^x-b^2e^{-x})^2}x^2dx = \frac{\pi^2}{2ab}$ $(ab>0)$

668. $\int_{-\infty}^\infty \frac{a^2e^x-b^2e^{-x}}{(a^2e^x+b^2e^{-x})^2}x^2dx = \frac{\pi}{ab}\ln\frac{b}{a}$ $(ab>0)$

669. $\int_0^\infty \left(\frac{1}{1-e^{-x}}-\frac{1}{x}\right)e^{-x}dx = \gamma$

[这里，γ 为欧拉常数(见附录)，以下同]

670. $\int_0^\infty \left(\frac{1}{1-e^{-x}}-\frac{1}{x}\right)\frac{dx}{x} = \gamma$

671. $\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x} = \gamma$

672. $\int_0^\infty \left(\frac{e^{-x}-1}{x}+\frac{1}{1+x}\right)\frac{dx}{x} = \gamma-1$

673. $\int_0^\infty \left(\frac{1}{2}-\frac{1}{1+e^{-x}}\right)e^{-2x}\frac{dx}{x} = \frac{1}{2}\ln\frac{\pi}{4}$

674. $\int_0^\infty \left(\frac{1}{2}e^{-2x}-\frac{1}{e^x+1}\right)\frac{dx}{x} = -\frac{1}{2}\ln\pi$

675. $\int_0^\infty \left(pe^{-x}-\frac{1-e^{-px}}{x}\right)\frac{dx}{x} = p\ln p-p$ $(p>0)$

676. $\int_0^\infty \left(e^{-x}-e^{-2x}-\frac{1}{x}e^{-2x}\right)\frac{dx}{x} = 1-\ln 2$

677. $\int_0^\infty \left(e^{-px} - \frac{1}{1+ax}\right)\frac{dx}{x} = \ln\frac{a}{p} - \gamma \quad (a>0,\ \mathrm{Re}\ p>0)$

678. $\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2}\right)\frac{dx}{x} = \ln\frac{a}{p} - \gamma \quad (p>0)$

Ⅱ.1.3.4 指数函数与有理函数组合的积分

679. $\int_0^\infty e^{-px}(x+b)^q dx = p^{-q-1}e^{bp}\Gamma(q+1,bp) \quad (|\arg b|<\pi,\ \mathrm{Re}\ p>0)$

[这里，$\Gamma(p,q)$为补余不完全伽马函数(见附录)，以下同]

680. $\int_0^\infty \frac{e^{-ax}}{b+x}dx = -e^{ab}\mathrm{Ei}(-ab) \quad (\mathrm{Re}\ a>0,\ |\arg b|<\pi)$

[这里，$\mathrm{Ei}(z)$为指数积分(见附录)，以下同]

681. $\int_0^\infty \frac{e^{-ax}}{b-x}dx = e^{-ab}\mathrm{Ei}(ab) \quad (\mathrm{Re}\ a>0,\ b>0)$

682. $\int_{-\infty}^\infty \frac{e^{ipx}}{x-b}dx = i\pi e^{ibp} \quad (p>0)$

683. $\int_0^\infty \frac{x^{q-1}e^{-px}}{x+b}dx = b^{q-1}e^{bp}\Gamma(q)\Gamma(1-q,bp)$

$(\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0,\ |\arg b|<\pi)$

[这里，$\Gamma(z)$为伽马函数(见附录)]

684. $\int_0^1 \frac{xe^x}{(1+x)^2}dx = \frac{e}{2} - 1$

685. $\int_0^\infty \frac{e^{-px}}{(a+x)^2}dx = pe^{ap}\mathrm{Ei}(-ap) + \frac{1}{a} \quad (a>0,\ p>0)$

686. $\int_{-\infty}^\infty \frac{e^{-ipx}}{a^2+x^2}dx = \frac{\pi}{a}e^{-|ap|} \quad (a>0,\ p\ 为实数)$

Ⅱ.1.3.5 指数函数与无理函数组合的积分

687. $\int_0^\infty \sqrt{x}e^{-qx}dx = \frac{1}{2q}\sqrt{\frac{\pi}{q}} \quad (q>0)$

688. $\int_0^\infty \frac{e^{-qx}}{\sqrt{x}}dx = \sqrt{\frac{\pi}{q}} \quad (q>0)$

689. $\int_0^u \frac{e^{-qx}}{\sqrt{x}}dx = \frac{\sqrt{\pi}}{q}\Phi(\sqrt{qu})$

[这里，$\Phi(x)$为概率积分(见附录)，以下同]

690. $\int_{-1}^\infty \frac{e^{-qx}}{\sqrt{1+x}}dx = e^q\sqrt{\frac{\pi}{q}} \quad (q>0)$

691. $\int_1^{\infty} \frac{e^{-qx}}{\sqrt{x-1}}dx = e^{-q}\sqrt{\frac{\pi}{q}}$ $(\mathrm{Re}\ q > 0)$

692. $\int_0^{\infty} \frac{e^{-qx}}{\sqrt{x+b}}dx = \sqrt{\frac{\pi}{q}}e^{bq}\left[1-\Phi\left(\sqrt{bq}\right)\right]$ $(\mathrm{Re}\ q > 0,\ |\arg b| < \pi)$

693. $\int_u^{\infty} \frac{e^{-qx}\sqrt{x-u}}{x}dx = \sqrt{\frac{\pi}{q}}e^{-qu} - \pi\sqrt{u}\left[1-\Phi\left(\sqrt{qu}\right)\right]$

$(u > 0,\ \mathrm{Re}\ q > 0)$

694. $\int_u^{\infty} \frac{e^{-qx}}{x\sqrt{x-u}}dx = \frac{\pi}{\sqrt{u}}\left[1-\Phi\left(\sqrt{qu}\right)\right]$ $(u > 0,\ \mathrm{Re}\ q > 0)$

695. $\int_{-1}^{1} \frac{e^{2x}}{\sqrt{1-x^2}}dx = \pi I_0(2)$

[这里，$I_\nu(z)$为第一类修正贝塞尔函数(见附录)，以下同]

696. $\int_0^{2} \frac{e^{-px}}{\sqrt{x(2-x)}}dx = \pi e^{-p}I_0(p)$ $(p > 0)$

697. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x(x+a)}}dx = e^{\frac{ap}{2}}K_0\left(\frac{ap}{2}\right)$ $(a > 0,\ p > 0)$

[这里，$K_\nu(z)$为第二类修正贝塞尔函数(见附录)]

Ⅱ.1.3.6 指数函数的代数函数与幂函数组合的积分

698. $\int_0^{\infty} xe^{-x}\sqrt{1-e^{-x}}dx = \frac{4}{3}\left(\frac{4}{3}-\ln 2\right)$

699. $\int_0^{\infty} xe^{-x}\sqrt{1-e^{-2x}}dx = \frac{\pi}{4}\left(\frac{1}{2}+\ln 2\right)$

700. $\int_0^{\infty} \frac{x}{\sqrt{e^x-1}}dx = 2\pi\ln 2$

701. $\int_0^{\infty} \frac{x^2}{\sqrt{e^x-1}}dx = 4\pi\left[(\ln 2)^2+\frac{\pi^2}{12}\right]$

702. $\int_0^{\infty} \frac{xe^{-x}}{\sqrt{e^x-1}}dx = \frac{\pi}{2}(2\ln 2-1)$

703. $\int_0^{\infty} \frac{xe^{-x}}{\sqrt{e^{2x}-1}}dx = 1-\ln 2$

704. $\int_0^{\infty} \frac{xe^{-2x}}{\sqrt{e^x-1}}dx = \frac{3\pi}{4}\left(\ln 2-\frac{7}{12}\right)$

Ⅱ.1.3.7 更复杂自变数的指数函数与幂函数组合的积分

705. $\int_0^{\infty} x^{2n}e^{-px^2}dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}}$ $(p > 0,\ n = 0,1,2,\cdots)$

706. $\int_0^\infty x^{2n+1}e^{-px^2}dx = \frac{n!!}{2p^{n+1}} \quad (p>0,\ n=0,1,2,\cdots)$

707. $\int_0^\infty (1+2bx^2)e^{-px^2}dx = \frac{p+b}{2}\sqrt{\frac{\pi}{p^3}} \quad (\mathrm{Re}\ p>0)$

708. $\int_0^\infty \left(e^{-x^2}-e^{-x}\right)\frac{dx}{x} = \frac{1}{2}\gamma$

[这里，γ 为欧拉常数(见附录)，以下同]

709. $\int_0^\infty \left(e^{-x^4}-e^{-x}\right)\frac{dx}{x} = \frac{3}{4}\gamma$

710. $\int_0^\infty \left(e^{-x^4}-e^{-x^2}\right)\frac{dx}{x} = \frac{1}{4}\gamma$

711. $\int_0^\infty \left(e^{-x^2}-\frac{1}{1+x^2}\right)\frac{dx}{x} = -\frac{1}{2}\gamma$

712. $\int_0^\infty \left(e^{-px^2}-e^{-qx^2}\right)\frac{dx}{x^2} = \sqrt{\pi}(\sqrt{q}-\sqrt{p}) \quad (\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

713. $\int_0^1 \frac{e^{x^2}-1}{x^2}dx = \sum_{k=1}^\infty \frac{1}{k!(2k-1)}$

714. $\int_0^\infty \left[\exp\left(-\frac{a}{x^2}\right)-1\right]e^{-px^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{p}}\left[\exp\left(-2\sqrt{ap}\right)-1\right]$

$(\mathrm{Re}\ a>0,\ \mathrm{Re}\ p>0)$

715. $\int_0^\infty x^2\exp\left(-\frac{a}{x^2}-px^2\right)dx = \frac{1}{4}\sqrt{\frac{\pi}{p^3}}(1+2\sqrt{ap})\exp(-2\sqrt{ap})$

$(\mathrm{Re}\ a>0,\ \mathrm{Re}\ p>0)$

716. $\int_0^\infty \frac{1}{x^2}\exp\left(-\frac{a}{x^2}-px^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp(-2\sqrt{ap}) \quad (\mathrm{Re}\ a>0,\ \mathrm{Re}\ p>0)$

717. $\int_0^\infty \left[\exp(-x^2)-\frac{1}{1+x^{2^{n+1}}}\right]\frac{dx}{x} = -\frac{1}{2^n}\gamma$

718. $\int_0^\infty \left[\exp\left(-x^{2^n}\right)-\frac{1}{1+x^2}\right]\frac{dx}{x} = -\frac{1}{2^n}\gamma$

719. $\int_0^\infty [\exp(-x^p)-\exp(-x^q)]\frac{dx}{x} = \frac{p-q}{pq}\gamma \quad (p>0,\ q>0)$

720. $\int_{-\infty}^\infty xe^x\exp(-ae^x)dx = -\frac{1}{a}(\ln a+\gamma) \quad (\mathrm{Re}\ a>0)$

721. $\int_{-\infty}^\infty xe^x\exp(-ae^{2x})dx = -\frac{1}{4}\sqrt{\frac{\pi}{a}}[\ln(4a)+\gamma] \quad (\mathrm{Re}\ a>0)$

Ⅱ.1.3.8 含有对数函数 $\ln x$ 和 $(\ln x)^n$ 的积分

722. $\int_0^1 (\ln x)^n dx = e^{n\pi i}\Gamma(n+1) = (-1)^n n! \quad (n>-1)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

723. $\int_0^1 x^m(\ln x)^n \mathrm{d}x = (-1)^n \frac{\Gamma(n+1)}{(m+1)^{m+1}}$ （$m>-1$，n 为正整数）

724. $\int_0^1 \ln\frac{1}{x}\mathrm{d}x = 1$

725. $\int_0^1 \left(\ln\frac{1}{x}\right)^n \mathrm{d}x = n!$

726. $\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1} \mathrm{d}x = \Gamma(p)$ （$\mathrm{Re}\ p>0$）

727. $\int_0^1 \left(\ln\frac{1}{x}\right)^{-p} \mathrm{d}x = \frac{\pi}{\Gamma(p)}\csc p\pi$ （$\mathrm{Re}\ p<1$）

728. $\int_0^1 x\ln\frac{1}{x}\mathrm{d}x = \frac{1}{4}$

729. $\int_0^1 x^m\left(\ln\frac{1}{x}\right)^n \mathrm{d}x = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$ （$m>-1$，$n>-1$）

730. $\int_0^1 \ln(1+x)\mathrm{d}x = 2\ln 2-1$

731. $\int_0^1 \ln(1-x)\mathrm{d}x = -1$

732. $\int_0^1 x\ln(1+x)\mathrm{d}x = \frac{1}{4}$

733. $\int_0^1 x\ln(1-x)\mathrm{d}x = -\frac{3}{4}$

734. $\int_0^1 \frac{\ln(1+x)}{x}\mathrm{d}x = \frac{\pi^2}{12}$

735. $\int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x = -\frac{\pi^2}{6}$

736. $\int_0^1 \frac{\ln(1+x^a)}{x}\mathrm{d}x = \frac{\pi^2}{12a}$ （$a\neq 0$）

737. $\int_0^1 \frac{\ln(1-x^a)}{x}\mathrm{d}x = -\frac{\pi^2}{6a}$ （$a\neq 0$）

738. $\int_0^1 \ln\frac{1+x}{1-x}\frac{\mathrm{d}x}{x} = \frac{\pi^2}{4}$

739. $\int_0^1 \frac{\mathrm{d}x}{a+\ln x} = \mathrm{e}^{-a}\mathrm{Ei}(a)$

[这里，$\mathrm{Ei}(z)$为指数积分(见附录)，以下同]

740. $\int_0^1 \frac{\mathrm{d}x}{a-\ln x} = -\mathrm{e}^{a}\mathrm{Ei}(a)$

741. $\int_0^1 \ln x\ln(1+x)\mathrm{d}x = 2-2\ln 2-\frac{\pi^2}{12}$

742. $\int_0^1 \ln x\ln(1-x)\mathrm{d}x = 2-\frac{\pi^2}{6}$

743. $\int_0^1 \sqrt{\ln \frac{1}{x}}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$

744. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi}$

745. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{\ln(-\ln x)}} = \sqrt{\pi}$

Ⅱ.1.3.9 含有更复杂自变数的对数函数的积分

746. $\int_0^\infty \ln \frac{a^2+x^2}{b^2+x^2}\mathrm{d}x = (a-b)\pi \quad (a>0,\ b>0)$

747. $\int_0^\infty \ln \frac{a^2-x^2}{b^2-x^2}\mathrm{d}x = (a+b)\pi$

748. $\int_0^\infty \ln x \ln \frac{a^2+x^2}{b^2+x^2}\mathrm{d}x = (b-a)\pi + \pi \ln \frac{a^a}{b^b} \quad (a>0,\ b>0)$

749. $\int_0^\infty \ln x \ln\left(1+\frac{b^2}{x^2}\right)\mathrm{d}x = \pi b(\ln b - 1) \quad (b>0)$

750. $\int_0^\infty \ln(a^2+x^2)\ln\left(1+\frac{b^2}{x^2}\right)\mathrm{d}x = 2\pi[(a+b)\ln(a+b) - a\ln a - b]$

$(a>0,\ b>0)$

751. $\int_0^\infty \ln\left(1+\frac{a^2}{x^2}\right)\ln\left(1+\frac{b^2}{x^2}\right)\mathrm{d}x = 2\pi[(a+b)\ln(a+b) - a\ln a - b\ln b]$

$(a>0,\ b>0)$

752. $\int_0^1 \ln\left(\ln \frac{1}{x}\right)\mathrm{d}x = -\gamma$

[这里，γ 为欧拉常数(见附录)，以下同]

753. $\int_0^1 \frac{\mathrm{d}x}{\ln\left(\ln \frac{1}{x}\right)} = 0$

754. $\int_0^1 \frac{\ln\left(\ln \frac{1}{x}\right)}{\sqrt{\ln \frac{1}{x}}}\mathrm{d}x = -(2\ln 2 + \gamma)\sqrt{\pi}$

755. $\int_0^{\frac{\pi}{4}} \ln(\sin x)\mathrm{d}x = -\frac{\pi}{4}\ln 2 - \frac{1}{2}G$

[这里，G 为卡塔兰常数(见附录)，以下同]

756. $\int_0^{\frac{\pi}{4}} \ln(\cos x)\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}G$

757. $\int_0^{\frac{\pi}{4}} \ln(\tan x)\mathrm{d}x = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\tan x)\mathrm{d}x = -G$

758. $\int_0^{\frac{\pi}{4}} \ln(\cot x)\mathrm{d}x = G$

759. $\int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\cos x)\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln(\sin x)\mathrm{d}x = -\frac{\pi}{2}\ln 2$

760. $\int_0^{\frac{\pi}{2}} \ln(\tan x)\mathrm{d}x = 0$

761. $\int_0^{\frac{\pi}{2}} \ln(\sec x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\csc x)\mathrm{d}x = \frac{\pi}{2}\ln 2$

762. $\int_0^{\frac{\pi}{2}} [\ln(\sin x)]^2\mathrm{d}x = \int_0^{\frac{\pi}{2}} [\ln(\cos x)]^2\mathrm{d}x = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right]$

763. $\int_0^{\frac{\pi}{2}} \ln(1 \pm \sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(1 \pm \cos x)\mathrm{d}x = -\frac{\pi}{2}\ln 2 \pm 2G$

764. $\int_0^{\pi} \ln(1 \pm \sin x)\mathrm{d}x = -\pi\ln 2 \pm 4G$

765. $\int_0^{\pi} \ln(1 \pm \cos x)\mathrm{d}x = -\pi\ln 2$

766. $\int_0^{\pi} \ln(a + b\cos x)\mathrm{d}x = \pi\ln\frac{a + \sqrt{a^2 - b^2}}{2} \quad (a \geqslant |b| > 0)$

767. $\int_0^{\pi} \ln(a + b\cos x)^2\mathrm{d}x = 2\pi\ln\frac{a + \sqrt{a^2 - b^2}}{2} \quad (a \geqslant |b| > 0)$

768. $\int_0^{\frac{\pi}{4}} \ln(\cos x + \sin x)\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\cos x + \sin x)\mathrm{d}x = -\frac{\pi}{8}\ln 2 + \frac{1}{2}G$

769. $\int_0^{\frac{\pi}{4}} \ln(\cos x - \sin x)\mathrm{d}x = -\frac{\pi}{8}\ln 2 - \frac{1}{2}G$

770. $\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)^2\mathrm{d}x = \begin{cases} -2\pi\ln 2 & (a^2 \leqslant 1) \\ 2\pi\ln\frac{a + \sqrt{a^2 - 1}}{2} & (a > 1) \end{cases}$

771. $\int_0^{\frac{\pi}{2}} \ln(1 + a\sin^2 x)\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln(1 + a\sin^2 x)\mathrm{d}x$

$= \int_0^{\frac{\pi}{2}} \ln(1 + a\cos^2 x)\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln(1 + a\cos^2 x)\mathrm{d}x$

$= \pi\ln\frac{1 + \sqrt{1 + a}}{2} \quad (a \geqslant -1)$

772. $\int_0^{\frac{\pi}{2}} \ln(a\tan x)\mathrm{d}x = \frac{\pi}{2}\ln a \quad (a > 0)$

773. $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x = \frac{\pi}{8}\ln 2$

774. $\int_0^{\frac{\pi}{2}} \ln(1 + \tan x)\mathrm{d}x = \frac{\pi}{4}\ln 2 + G$

775. $\int_0^{\frac{\pi}{4}} \ln(1 - \tan x)\mathrm{d}x = \frac{\pi}{8}\ln 2 - G$

776. $\int_0^{\frac{\pi}{2}} \ln(1-\tan x)^2 \mathrm{d}x = \frac{\pi}{2}\ln 2 - 2G$

777. $\int_0^{\frac{\pi}{4}} \ln(1+\cot x)\mathrm{d}x = \frac{\pi}{8}\ln 2 + G$

778. $\int_0^{\frac{\pi}{4}} \ln(\cot x - 1)\mathrm{d}x = \frac{\pi}{8}\ln 2$

779. $\int_0^{\infty} \ln(1+\mathrm{e}^{-x})\mathrm{d}x = \frac{\pi^2}{12}$

780. $\int_0^{\infty} \ln(1-\mathrm{e}^{-x})\mathrm{d}x = -\frac{\pi^2}{6}$

781. $\int_0^{\infty} \ln\frac{\mathrm{e}^x+1}{\mathrm{e}^x-1}\mathrm{d}x = \frac{\pi^2}{4}$

782. $\int_0^{\infty} \ln(1+2\mathrm{e}^{-x}\cos t + \mathrm{e}^{-2x})\mathrm{d}x = \frac{\pi^2}{6} - \frac{t^2}{2} \quad (|t| < \pi)$

Ⅱ.1.3.10 对数函数与有理函数组合的积分

783. $\int_0^1 \frac{\ln x}{1+x}\mathrm{d}x = -\frac{\pi^2}{12}$

784. $\int_0^1 \frac{\ln x}{1-x}\mathrm{d}x = -\frac{\pi^2}{6}$

785. $\int_0^1 \frac{\ln x}{1+x^2}\mathrm{d}x = -\int_1^{\infty} \frac{\ln x}{1+x^2}\mathrm{d}x = -G$

[这里，G 为卡塔兰常数(见附录)，以下同]

786. $\int_0^{\infty} \frac{\ln x}{1+x^2}\mathrm{d}x = 0$

787. $\int_0^1 \frac{\ln x}{1-x^2}\mathrm{d}x = -\frac{\pi^2}{8}$

788. $\int_0^{\infty} \frac{\ln x}{1-x^2}\mathrm{d}x = -\frac{\pi^2}{4}$

789. $\int_0^1 \frac{\ln x}{(1+x)^2}\mathrm{d}x = -\ln 2$

790. $\int_0^1 \frac{x\ln x}{1+x^2}\mathrm{d}x = -\frac{\pi^2}{48}$

791. $\int_0^1 \frac{x\ln x}{1-x^2}\mathrm{d}x = -\frac{\pi^2}{24}$

792. $\int_0^1 \frac{x^p - x^q}{\ln x}\mathrm{d}x = \ln\frac{p+1}{q+1} \quad (p > -1,\ q > -1)$

793. $\int_0^a \frac{\ln x}{a^2+x^2}\mathrm{d}x = \frac{\pi\ln a}{4a} - \frac{G}{a} \quad (a > 0)$

794. $\int_0^\infty \frac{\ln x}{a^2+x^2}\mathrm{d}x = \frac{\pi}{2a}\ln a$

795. $\int_0^\infty \frac{\ln x}{a^2+b^2x^2}\mathrm{d}x = \frac{\pi}{2ab}\ln\frac{a}{b}$

796. $\int_0^\infty \frac{\ln x}{a^2-b^2x^2}\mathrm{d}x = -\frac{\pi}{4ab}$

797. $\int_0^\infty \frac{\ln(x+1)}{1+x^2}\mathrm{d}x = \frac{\pi\ln 2}{4}+G$

798. $\int_0^\infty \frac{\ln(x-1)}{1+x^2}\mathrm{d}x = \frac{\pi\ln 2}{8}$

799. $\int_0^\infty \frac{\ln x}{(a+x)(b+x)}\mathrm{d}x = \frac{(\ln a)^2-(\ln b)^2}{2(a-b)} \quad (a\neq b)$

Ⅱ.1.3.11 对数函数与无理函数组合的积分

800. $\int_0^{\frac{\sqrt{2}}{2}} \frac{\ln x}{\sqrt{1-x^2}}\mathrm{d}x = -\frac{\pi}{4}\ln 2-\frac{1}{2}G$

[这里，G 为卡塔兰常数(见附录)]

801. $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\mathrm{d}x = -\frac{\pi}{2}\ln 2$

802. $\int_0^1 \sqrt{1-x^2}\ln x\mathrm{d}x = -\frac{\pi}{8}-\frac{\pi}{4}\ln 2$

803. $\int_0^1 x\sqrt{1-x^2}\ln x\mathrm{d}x = \frac{1}{3}\ln 2-\frac{4}{9}$

804. $\int_0^1 \frac{\ln x}{\sqrt{x(1-x^2)}}\mathrm{d}x = -\frac{\sqrt{2\pi}}{8}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$

[这里，$\Gamma(z)$ 为伽马函数(见附录)]

805. $\int_0^\infty \frac{\ln x}{x^2\sqrt{x^2-1}}\mathrm{d}x = 1-\ln 2$

Ⅱ.1.3.12 对数函数与幂函数和有理函数组合的积分

806. $\int_0^\infty \frac{x^{p-1}\ln x}{a+x}\mathrm{d}x = \frac{\pi a^{p-1}}{\sin p\pi}(\ln a-\pi\cot p\pi) \quad (0<\mathrm{Re}\ p<1,\ |\arg a|<\pi)$

807. $\int_0^\infty \frac{x^{p-1}\ln x}{a-x}\mathrm{d}x = \pi a^{p-1}\left(\cot p\pi\ln a-\frac{\pi}{\sin^2 p\pi}\right) \quad (0<\mathrm{Re}\ p<1,\ a>0)$

808. $\int_0^1 \frac{x^{2n}\ln x}{1+x}\mathrm{d}x = -\frac{\pi^2}{12}+\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k^2}$

809. $\int_0^1 \frac{x^{2n-1}\ln x}{1+x}\mathrm{d}x = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1}\frac{(-1)^k}{k^2}$

810. $\int_0^\infty \frac{x^{p-1}\ln x}{1+x^2}\mathrm{d}x = -\frac{\pi^2}{4}\csc\frac{p\pi}{2}\cot\frac{p\pi}{2} \quad (0<p<2)$

811. $\int_0^\infty \frac{x^{p-1}\ln x}{1-x^2}\mathrm{d}x = -\frac{\pi^2}{4}\csc^2\frac{p\pi}{2} \quad (0<p<2)$

812. $\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\mathrm{d}x = -\frac{\pi^2}{q^2}\frac{\cos\frac{p\pi}{q}}{\sin^2\frac{p\pi}{q}} \quad (0<p<q)$

813. $\int_0^\infty \frac{x^{p-1}\ln x}{1-x^q}\mathrm{d}x = -\frac{\pi^2}{q^2\sin^2\frac{p\pi}{q}} \quad (0<p<q)$

814. $\int_0^1 \frac{x^{q-1}\ln x}{1-x^{2q}}\mathrm{d}x = -\frac{\pi^2}{8q^2} \quad (q>0)$

815. $\int_0^\infty \frac{x^{p-1}\ln x}{(x+a)^2}\mathrm{d}x = \frac{(1-p)a^{p-2}\pi}{\sin p\pi}\left(\ln a - \pi\cot p\pi + \frac{1}{p-1}\right)$
$(a>0, 0<\mathrm{Re}\ p<2, p\neq 1)$

816. $\int_1^\infty (x-1)^{p-1}\ln x\mathrm{d}x = \frac{\pi}{p}\csc p\pi \quad (-1<p<0)$

817. $\int_0^\infty \frac{\ln x}{x^p(x^q-1)}\mathrm{d}x = \frac{\pi^2}{q^2\sin^2\frac{(p-1)\pi}{q}} \quad (p<1,\ p+q>1)$

818. $\int_0^\infty \frac{1-x^p}{1-x^2}\ln x\mathrm{d}x = \frac{\pi^2}{4}\tan^2\frac{p\pi}{2} \quad (p<1)$

Ⅱ.1.3.13 含有对数函数的幂函数的积分

819. $\int_0^1 \frac{(\ln x)^2}{1+2x\cos t+x^2}\mathrm{d}x = \frac{t(\pi^2-t^2)}{6\sin t} \quad (0<t<\pi)$

820. $\int_0^1 \frac{(\ln x)^3}{1+x}\mathrm{d}x = -\frac{7\pi^4}{120}$

821. $\int_0^1 \frac{(\ln x)^3}{1-x}\mathrm{d}x = -\frac{\pi^4}{15}$

822. $\int_1^\infty \frac{(\ln x)^p}{x^2}\mathrm{d}x = \Gamma(1+p) \quad (p>-1)$
[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

823. $\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1}x^{q-1}\mathrm{d}x = \frac{1}{q^p}\Gamma(p) \quad (\mathrm{Re}\ p>0,\ \mathrm{Re}\ q>0)$

824. $\int_0^1 x^{p-1}\sqrt{\ln\frac{1}{x}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{p^3}} \quad (p>0)$

825. $\int_0^1 \frac{x^{p-1}}{\sqrt{\ln\frac{1}{x}}}dx = \sqrt{\frac{\pi}{p}} \quad (p>0)$

Ⅱ.1.3.14 更复杂自变数的对数函数与代数函数组合的积分

826. $\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{x}dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$

827. $\int_0^1 \frac{\ln\left(1-\frac{x}{2}\right)}{x}dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$

828. $\int_0^1 \frac{\ln(1+x)}{1+x}dx = \frac{1}{2}(\ln 2)^2$

829. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$

830. $\int_0^\infty \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{4}\ln 2 + G$

[这里，G 为卡塔兰常数(见附录)，以下同]

831. $\int_0^1 \frac{\ln(1-x)}{1+x^2}dx = \frac{\pi}{8}\ln 2 - G$

832. $\int_1^\infty \frac{\ln(1-x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$

833. $\int_0^1 \frac{\ln(1+x)}{x(1+x)}dx = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2$

834. $\int_0^\infty \frac{\ln(1+x)}{x(1+x)}dx = \frac{\pi^2}{6}$

835. $\int_0^1 \frac{\ln(1\pm x)}{\sqrt{1-x^2}}dx = -\frac{\pi}{2}\ln 2 \pm 2G$

836. $\int_0^1 \frac{x\ln(1\pm x)}{\sqrt{1-x^2}}dx = -1 \pm \frac{\pi}{2}$

837. $\int_0^1 \frac{\ln(1+x^2)}{x^2}dx = \frac{\pi}{2} - \ln 2$

838. $\int_0^\infty \frac{\ln(1+x^2)}{x^2}dx = \pi$

839. $\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx = \frac{\pi}{2}\ln 2 - G$

840. $\int_1^\infty \frac{\ln(1+x^2)}{1+x^2}dx = \frac{\pi}{2}\ln 2 + G$

841. $\int_0^1 \frac{\ln(1+ax^2)}{\sqrt{1-x^2}}dx = \pi\ln\frac{1+\sqrt{1+a}}{2} \quad (a \geqslant -1)$

842. $\int_0^1 \frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}}\mathrm{d}x = \pi\ln\frac{1+\sqrt{1-a^2}}{2} \quad (a^2<1)$

843. $\int_0^1 \ln\frac{1+x^2}{x}\cdot\frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{2}\ln 2$

844. $\int_0^\infty \ln\frac{1+x^2}{x}\cdot\frac{\mathrm{d}x}{1+x^2} = \pi\ln 2$

845. $\int_0^1 \ln\frac{1-x^2}{x}\cdot\frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{4}\ln 2$

846. $\int_0^\infty \ln\frac{1+x^2}{x}\cdot\frac{\mathrm{d}x}{1-x^2} = 0$

847. $\int_0^1 \ln\frac{1+x^2}{x+1}\cdot\frac{\mathrm{d}x}{1+x^2} = \frac{3\pi}{8}\ln 2 - G$

848. $\int_1^\infty \ln\frac{1+x^2}{x+1}\cdot\frac{\mathrm{d}x}{1+x^2} = \frac{3\pi}{8}\ln 2$

849. $\int_0^1 \ln\frac{1+x^2}{1-x}\cdot\frac{\mathrm{d}x}{1+x^2} = \frac{3\pi}{8}\ln 2$

850. $\int_1^\infty \ln\frac{1+x^2}{x-1}\cdot\frac{\mathrm{d}x}{1+x^2} = \frac{3\pi}{8}\ln 2 + G$

851. $\int_0^\infty \ln\frac{1+x^2}{x^2}\cdot\frac{x}{1+x^2}\mathrm{d}x = \frac{\pi^2}{12}$

852. $\int_0^\pi \ln(\sin x)x\mathrm{d}x = \frac{1}{2}\int_0^\pi \ln(\cos^2 x)x\mathrm{d}x = -\frac{\pi^2}{2}\ln 2$

853. $\int_0^\infty \frac{\ln(\sin ax)}{b^2+x^2}\mathrm{d}x = \frac{\pi}{2b}\ln\frac{\sinh ab}{\mathrm{e}^{ab}} \quad (a\neq 0,\ b\neq 0)$

854. $\int_0^\infty \frac{\ln(\cos ax)}{b^2+x^2}\mathrm{d}x = \frac{\pi}{2b}\ln\frac{\cosh ab}{\mathrm{e}^{ab}} \quad (a\neq 0,\ b\neq 0)$

855. $\int_0^\infty \frac{\ln(\sin ax)}{b^2-x^2}\mathrm{d}x = \frac{a\pi}{2} - \frac{\pi^2}{4b} \quad (a\neq 0,\ b\neq 0)$

856. $\int_0^\infty \frac{\ln(\cos ax)}{b^2-x^2}\mathrm{d}x = \frac{a\pi}{2}$

Ⅱ.1.3.15 对数函数与指数函数组合的积分

857. $\int_0^\infty \mathrm{e}^{-px}\ln x\,\mathrm{d}x = -\frac{1}{p}(\ln p+\gamma) \quad (\mathrm{Re}\ p>0)$

[这里，γ 为欧拉常数(见附录)，以下同]

858. $\int_0^\infty \mathrm{e}^{-px}(\ln x)^2\mathrm{d}x = \frac{1}{p}\left[\frac{\pi^2}{6}+(\ln p+\gamma)^2\right] \quad (\mathrm{Re}\ p>0)$

859. $\int_0^\infty \mathrm{e}^{-ax}\ln\frac{1}{x}\mathrm{d}x = \frac{\ln a+\gamma}{a} \quad (a>0)$

860. $\int_0^{\infty} x e^{-ax} \ln \frac{1}{x} dx = \frac{2(\ln a + \gamma - 1)}{a^2} \quad (a > 0)$

861. $\int_0^{\infty} \frac{e^{-ax}}{\ln x} dx = 0 \quad (a > 0)$

862. $\int_0^{\infty} e^{-x} \ln x \, dx = -\gamma$

863. $\int_0^{\infty} e^{-x^2} \ln x \, dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$

864. $\int_0^{\infty} e^{-x^2} (\ln x)^2 dx = \frac{\sqrt{\pi}}{8}\left[\frac{\pi^2}{2} + (2\ln 2 + \gamma)^2\right]$

865. $\int_0^1 x e^x \ln(1 - x) dx = 1 - e$

866. $\int_0^1 (1 - x) e^{-x} \ln x \, dx = \frac{1 - e}{e}$

Ⅱ.1.3.16 对数函数与三角函数组合的积分

867. $\int_0^{\frac{\pi}{2}} \ln(\sin x) \sin x dx = \ln 2 - 1$

868. $\int_0^{\frac{\pi}{2}} \ln(\sin x) \cos x dx = -1$

869. $\int_0^{\frac{\pi}{2}} \ln(\sin x) \cos 2nx dx = -\frac{\pi}{4n}$

870. $\int_0^{\frac{\pi}{2}} \ln(\sin x) \sin^2 x dx = \frac{\pi}{8}(1 - \ln 4)$

871. $\int_0^{\frac{\pi}{2}} \ln(\sin x) \cos^2 x dx = -\frac{\pi}{8}(1 + \ln 4)$

872. $\int_0^{\frac{\pi}{2}} \ln(\sin x) \tan x dx = -\frac{\pi^2}{24}$

873. $\int_0^{\frac{\pi}{2}} \ln(\sin 2x) \sin x dx = \int_0^{\frac{\pi}{2}} \ln(\sin 2x) \cos x dx = 2(\ln 2 - 1)$

874. $\int_0^{\frac{\pi}{2}} \ln(\tan x) \sin x dx = \ln 2$

875. $\int_0^{\frac{\pi}{2}} \ln(\tan x) \cos x dx = -\ln 2$

876. $\int_0^{\frac{\pi}{2}} \ln(\tan x) \sin^2 x dx = -\int_0^{\frac{\pi}{2}} \ln(\tan x) \cos^2 x dx = \frac{\pi}{4}$

877. $\int_0^{\frac{\pi}{2}} \ln\left(\cot \frac{x}{2}\right) \sin x dx = \ln 2$

878. $\int_0^1 \ln(\sin\pi x)\cos 2n\pi x\,\mathrm{d}x = 2\int_0^{\frac{1}{2}} \ln(\sin\pi x)\cos 2n\pi x\,\mathrm{d}x = \begin{cases} -\ln 2 & (n=0) \\ -\dfrac{1}{2n} & (n>0) \end{cases}$

879. $\int_0^1 \ln(\sin\pi x)\cos(2n+1)\pi x\,\mathrm{d}x = 0$

880. $\int_0^\infty \ln x \sin ax^2\mathrm{d}x = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left[\ln(4a)+\gamma-\frac{\pi}{2}\right] \quad (a>0)$

[这里，γ 为欧拉常数(见附录)，以下同]

881. $\int_0^\infty \ln x \cos ax^2\mathrm{d}x = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left[\ln(4a)+\gamma+\frac{\pi}{2}\right] \quad (a>0)$

882. $\int_0^\infty \ln\left|\frac{x+a}{x-a}\right|\sin bx\mathrm{d}x = \frac{\pi}{b}\sin ab \quad (a>0,\ b>0)$

883. $\int_0^\infty \ln\frac{a^2+x^2}{b^2+x^2}\cos cx\mathrm{d}x = \frac{\pi}{c}(\mathrm{e}^{-bc}-\mathrm{e}^{-ac}) \quad (a>0,\ b>0,\ c>0)$

884. $\int_0^\infty \ln(1+\mathrm{e}^{-ax})\cos bx\mathrm{d}x = \frac{a}{2b^2}-\frac{\pi}{2b\sinh\frac{b\pi}{a}} \quad (\mathrm{Re}\ a>0,\ b>0)$

885. $\int_0^\infty \ln(1-\mathrm{e}^{-ax})\cos bx\mathrm{d}x = \frac{a}{2b^2}-\frac{\pi}{2b}\coth\frac{b\pi}{a} \quad (\mathrm{Re}\ a>0,\ b>0)$

886. $\int_0^{\frac{\pi}{2}} \frac{\ln(1+p\sin x)}{\sin x}\mathrm{d}x = \frac{\pi^2}{8}-\frac{1}{2}(\arccos p)^2 \quad (p^2<1)$

887. $\int_0^{\frac{\pi}{2}} \frac{\ln(1+p\cos x)}{\cos x}\mathrm{d}x = \frac{\pi^2}{8}-\frac{1}{2}(\arccos p)^2 \quad (p^2<1)$

888. $\int_0^{\pi} \frac{\ln(1+p\cos x)}{\cos x}\mathrm{d}x = \pi\arcsin p \quad (p^2<1)$

889. $\int_0^{\frac{\pi}{4}} \frac{\ln(\tan x)}{\cos 2x}\mathrm{d}x = -\frac{\pi^2}{8}$

890. $\int_0^{\pi} \frac{\ln(\sin x)}{a+b\cos x}\mathrm{d}x = \frac{\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \quad (a>0,\ a>b)$

891. $\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)\sin x}{\sqrt{1+\sin^2 x}}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\ln(\cos x)\cos x}{\sqrt{1+\cos^2 x}}\mathrm{d}x = -\frac{\pi}{8}\ln 2$

Ⅱ.1.3.17 对数函数与三角函数、指数函数、双曲函数和幂函数组合的积分

892. $\int_0^\infty \ln x\,\frac{\sin ax}{x}\mathrm{d}x = -\frac{\pi}{2}(\ln a+\gamma) \quad (a>0)$

[这里，γ 为欧拉常数(见附录)，以下同]

893. $\int_0^\infty \ln x\,\frac{\cos ax-\cos bx}{x}\mathrm{d}x = \ln\frac{a}{b}\left[\frac{1}{2}\ln(ab)+\gamma\right] \quad (a>0,\ b>0)$

894. $\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}[(a-b)(\gamma-1) + a\ln a - b\ln b]$

$(a > 0,\ b > 0)$

895. $\int_0^\infty \ln x \frac{\sin^2 ax}{x^2} dx = -\frac{a\pi}{2}[\ln(2a) + \gamma - 1] \quad (a > 0)$

896. $\int_0^1 \frac{x(1+x)}{\ln x} \sin(\ln x) dx = \frac{\pi}{4}$

897. $\int_0^\infty e^{-ax} \ln x \sin bx dx = \frac{b}{a^2+b^2}\left(\ln\sqrt{a^2+b^2} + \frac{a}{b}\arctan\frac{b}{a} - \gamma\right)$

898. $\int_0^\infty e^{-ax} \ln x \cos bx dx = -\frac{a}{a^2+b^2}\left(\ln\sqrt{a^2+b^2} + \frac{a}{b}\arctan\frac{b}{a} + \gamma\right)$

899. $\int_0^\infty \frac{\ln x}{\cosh x} dx = \pi \ln \frac{\sqrt{2\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$

[这里，$\Gamma(z)$ 为伽马函数(见附录)]

900. $\int_0^\infty \frac{\ln(1+x^2)}{\cosh\frac{\pi x}{2}} dx = 2\ln\frac{4}{\pi}$

901. $\int_0^\infty \ln(\cos^2 t + e^{-2x}\sin^2 t) \frac{dx}{\sinh x} = -2t^2$

Ⅱ.1.4 双曲函数和反双曲函数的定积分

Ⅱ.1.4.1 含有 $\sinh ax$ 和 $\cosh bx$ 的积分，积分区间为 $[0, \infty)$

902. $\int_0^\infty \frac{dx}{\sinh ax} = \infty$

903. $\int_0^\infty \frac{x}{\sinh ax} dx = \frac{\pi^2}{4a^2} \quad (a > 0)$

904. $\int_0^\infty \frac{x^3}{\sinh ax} dx = \frac{\pi^4}{8a^4} \quad (a > 0)$

905. $\int_0^\infty \frac{x^m}{\sinh ax} dx = \frac{(2^{m+1}-1)m!}{2^m a^{m+1}} \sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{m+1}} \quad (a > 0)$

906. $\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a} \quad (a > 0)$

907. $\int_0^\infty \frac{x}{\cosh ax} dx = \frac{2G}{a^2} = \frac{1.831329803\cdots}{a^2} \quad (a > 0)$

[这里，G 为卡塔兰常数(见附录)]

908. $\int_0^\infty \frac{x^2}{\cosh ax}\mathrm{d}x = \frac{\pi^3}{8a^3} \quad (a>0)$

909. $\int_0^\infty \frac{x^m}{\cosh ax}\mathrm{d}x = \frac{2^{m+1}m!}{2^m a^{m+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^{m+1}} \quad (a>0)$

910. $\int_0^\infty \frac{\mathrm{d}x}{a+\sinh bx} = \frac{1}{b\sqrt{1+a^2}}\ln\frac{1+a+\sqrt{1+a^2}}{1+a-\sqrt{1+a^2}} \quad (1+a>\sqrt{1+a^2})$

911. $\int_0^\infty \frac{\mathrm{d}x}{a+\cosh bx} = \frac{1}{b\sqrt{a^2-1}}\ln\frac{1+a+\sqrt{a^2-1}}{1+a-\sqrt{a^2-1}} \quad (a^2>1)$

912. $\int_0^\infty \frac{\mathrm{d}x}{a\sinh cx+b\cosh cx} = \frac{1}{c\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \quad (a^2>b^2)$

913. $\int_0^\infty \frac{\mathrm{d}x}{a\sinh x+b\cosh x} = \begin{cases} \frac{2}{\sqrt{b^2-a^2}}\arctan\frac{\sqrt{b^2-a^2}}{a+b} & (b^2>a^2) \\ \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} & (a^2>b^2) \end{cases}$

914. $\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)\cosh \pi x} = 2-\frac{\pi}{2}$

915. $\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)\cosh\frac{\pi x}{2}} = \ln 2$

916. $\int_0^\infty \frac{x}{(1+x^2)\sinh \pi x}\mathrm{d}x = \ln 2-\frac{1}{2}$

917. $\int_0^\infty \frac{x}{(1+x^2)\sinh\frac{\pi x}{2}}\mathrm{d}x = \frac{\pi}{2}-1$

918. $\int_0^\infty \left(\frac{1}{\sinh x}-\frac{1}{x}\right)\frac{\mathrm{d}x}{x} = -\ln 2$

919. $\int_0^\infty \frac{\sinh ax}{\sinh bx}\mathrm{d}x = \frac{\pi}{2b}\tan\frac{a\pi}{2|b|} \quad (b\neq 0)$

920. $\int_0^\infty \frac{\cosh ax}{\cosh bx}\mathrm{d}x = \frac{\pi}{2b}\sec\frac{a\pi}{2b} \quad (b\neq 0)$

921. $\int_0^\infty \frac{\cosh 2ax}{\cosh^{2q} bx}\mathrm{d}x = \frac{4^{q-1}}{b}\mathrm{B}\left(q+\frac{a}{b}, q-\frac{a}{b}\right) \quad [\mathrm{Re}(q\pm a)>0,\ a>0,\ b>0]$

[这里，$\mathrm{B}(p,q)$ 为贝塔函数(见附录)，以下同]

922. $\int_0^\infty \frac{\sinh^p x}{\cosh^q x}\mathrm{d}x = \frac{1}{2}\mathrm{B}\left(\frac{p+1}{2}, \frac{q-p}{2}\right) \quad [\mathrm{Re}\ p>-1,\ \mathrm{Re}\ (p-q)<0]$

923. $\int_0^\infty x\frac{\sinh ax}{\cosh bx}\mathrm{d}x = \frac{\pi^2}{4b^2}\sin\frac{a\pi}{2b}\sec^2\frac{a\pi}{2b} \quad (b>|a|)$

924. $\int_0^\infty x^{2m+1}\frac{\sinh ax}{\cosh bx}\mathrm{d}x = \frac{\pi}{2b}\cdot\frac{\mathrm{d}^{2m+1}}{\mathrm{d}a^{2m+1}}\sec\frac{a\pi}{2b} \quad (b>|a|)$

925. $\int_0^{\infty} x\frac{\cosh ax}{\sinh bx}\mathrm{d}x = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^2 \quad (b > |a|)$

926. $\int_0^{\infty} x^{2m+1}\frac{\cosh ax}{\sinh bx}\mathrm{d}x = \frac{\pi}{2b}\cdot\frac{\mathrm{d}^{2m+1}}{\mathrm{d}a^{2m+1}}\tan\frac{a\pi}{2b} \quad (b > |a|)$

927. $\int_0^{\infty} x^2\frac{\sinh ax}{\sinh bx}\mathrm{d}x = \frac{\pi^3}{4b^3}\sin\frac{a\pi}{2b}\sec^3\frac{a\pi}{2b} \quad (b > |a|)$

928. $\int_0^{\infty} x^{2m}\frac{\sinh ax}{\sinh bx}\mathrm{d}x = \frac{\pi}{2b}\cdot\frac{\mathrm{d}^{2m}}{\mathrm{d}a^{2m}}\tan\frac{a\pi}{2b} \quad (b > |a|)$

929. $\int_0^{\infty} x^2\frac{\cosh ax}{\cosh bx}\mathrm{d}x = \frac{\pi^3}{8b^3}\left(2\sec^3\frac{a\pi}{2b} - \sec\frac{a\pi}{2b}\right) \quad (b > |a|)$

930. $\int_0^{\infty} x^{2m}\frac{\cosh ax}{\cosh bx}\mathrm{d}x = \frac{\pi}{2b}\cdot\frac{\mathrm{d}^{2m}}{\mathrm{d}a^{2m}}\sec\frac{a\pi}{2b} \quad (b > |a|)$

931. $\int_0^{\infty}\frac{\sinh ax}{\sinh\pi x}\cdot\frac{\mathrm{d}x}{1+x^2} = -\frac{a}{2}\cos a + \frac{1}{2}\sin a\ln[2(1+\cos a)] \quad (|a| \leqslant \pi)$

932. $\int_0^{\infty}\frac{\sinh ax}{\cosh\pi x}\cdot\frac{x\mathrm{d}x}{1+x^2} = -2\sin\frac{a}{2} + \frac{\pi}{2}\sin a - \cos a\ln\left(\tan\frac{a+\pi}{4}\right) \quad (|a| < \pi)$

933. $\int_0^{\infty}\frac{\cosh ax}{\sinh\pi x}\cdot\frac{x\mathrm{d}x}{1+x^2} = \frac{1}{2}(a\sin a - 1) + \frac{1}{2}\cos a\ln[2(1+\cos a)] \quad (|a| < \pi)$

934. $\int_0^{\infty}\frac{\cosh ax}{\cosh\pi x}\cdot\frac{\mathrm{d}x}{1+x^2} = 2\cos\frac{a}{2} - \frac{\pi}{2}\cos a - \sin a\ln\left(\tan\frac{a+\pi}{4}\right) \quad (|a| < \pi)$

935. $\int_0^{\infty}\frac{\sinh ax}{\cosh bx}\cdot\frac{\mathrm{d}x}{x} = \ln\left[\tan\left(\frac{a\pi}{4b} + \frac{\pi}{4}\right)\right] \quad (b > |a|)$

936. $\int_0^{\infty}\frac{\sinh^2 ax}{\sinh bx}\cdot\frac{\mathrm{d}x}{x} = \frac{1}{2}\ln\left(\sec\frac{a\pi}{b}\right) \quad (b > |2a|)$

937. $\int_0^{\infty}\frac{x}{\cosh^2 ax}\mathrm{d}x = \frac{\ln 2}{a^2} \quad (a \neq 0)$

938. $\int_{-\infty}^{\infty}\frac{x^2}{\sinh^2 x}\mathrm{d}x = \frac{\pi^2}{3}$

939. $\int_0^{\infty} x\frac{\sinh ax}{\cosh^2 ax}\mathrm{d}x = \frac{\pi}{2a^2} \quad (a > 0)$

940. $\int_0^{\infty} x^2\frac{\cosh ax}{\sinh^2 ax}\mathrm{d}x = \frac{\pi^2}{2a^3} \quad (a > 0)$

941. $\int_0^{\infty} x^2\frac{\sinh ax}{\cosh^2 ax}\mathrm{d}x = \frac{\ln 2}{2a^3} \quad (a \neq 0)$

942. $\int_0^{\infty}\frac{\tan\frac{x}{2}}{\cosh x}\frac{\mathrm{d}x}{x} = \ln 2$

943. $\int_0^{\infty}\left(\frac{1}{\sinh x} - \frac{1}{x}\right)\frac{\mathrm{d}x}{x} = -\ln 2$

944. $\int_{-\infty}^{\infty}\left(1 - \frac{\sqrt{2}\cosh x}{\sqrt{\cosh 2x}}\right)\mathrm{d}x = -\ln 2$

945. $\int_0^\infty \frac{\sin ax}{\sinh bx}\mathrm{d}x = \frac{\pi}{2b}\tanh\frac{a\pi}{2|b|} \quad (b \neq 0)$

946. $\int_0^\infty \frac{\cos ax}{\cosh bx}\mathrm{d}x = \frac{\pi}{2b}\operatorname{sech}\frac{a\pi}{2b} \quad (b \neq 0)$

947. $\int_0^\infty \frac{x\sin ax}{\cosh bx}\mathrm{d}x = \frac{\pi^2}{(2b)^2}\frac{\tanh\frac{a\pi}{2b}}{\cosh\frac{a\pi}{2b}} \quad (b \neq 0)$

948. $\int_0^\infty \frac{x\cos ax}{\sinh bx}\mathrm{d}x = \frac{\pi^2}{(2b)^2}\operatorname{sech}^2\frac{a\pi}{2b} \quad (b \neq 0)$

949. $\int_0^\infty \frac{\mathrm{d}x}{\cosh ax + \cos t} = \frac{t}{a}\csc t \quad (a > 0,\ 0 < t < \pi)$

Ⅱ.1.4.2 双曲函数与指数函数组合的积分

950. $\int_0^\infty \mathrm{e}^{-px}\sinh^q rx\mathrm{d}x = \frac{1}{2^{q+1}r}\mathrm{B}\left(\frac{p}{2r} - \frac{q}{2}, q+1\right)$

$(\mathrm{Re}\ r > 0,\ \mathrm{Re}\ q > 0,\ \mathrm{Re}\ p > \mathrm{Re}\ qr)$

[这里，$\mathrm{B}(p,q)$为贝塔函数(见附录)]

951. $\int_0^\infty \mathrm{e}^{-ax}\sinh bx\mathrm{d}x = \frac{b}{a^2 - b^2} \quad (|b| < a)$

952. $\int_0^\infty \mathrm{e}^{-ax}\cosh bx\mathrm{d}x = \frac{a}{a^2 - b^2} \quad (|b| < a)$

953. $\int_0^\infty \mathrm{e}^{-px^2}\sinh ax\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{p}}\exp\left(\frac{a^2}{4p}\right)\Phi\left(\frac{a}{2\sqrt{p}}\right) \quad (\mathrm{Re}\ p > 0)$

[这里，$\Phi(x)$为概率积分(见附录)]

954. $\int_0^\infty \mathrm{e}^{-px^2}\cosh ax\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{p}}\exp\left(\frac{a^2}{4p}\right) \quad (\mathrm{Re}\ p > 0)$

955. $\int_0^\infty \mathrm{e}^{-px^2}\sinh^2 ax\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{p}}\left[\exp\left(\frac{a^2}{p}\right) - 1\right] \quad (\mathrm{Re}\ p > 0)$

956. $\int_0^\infty \mathrm{e}^{-px^2}\cosh^2 ax\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{p}}\left[\exp\left(\frac{a^2}{p}\right) + 1\right] \quad (\mathrm{Re}\ p > 0)$

957. $\int_{-\infty}^\infty \mathrm{e}^{-px}\frac{\sinh px}{\sinh qx}\mathrm{d}x = \frac{\pi}{2q}\tan\frac{p\pi}{q} \quad (\mathrm{Re}\ q > 2|\mathrm{Re}\ p|)$

958. $\int_0^\infty \mathrm{e}^{-qx}\frac{\sinh px}{\sinh qx}\mathrm{d}x = \frac{1}{p} - \frac{\pi}{2q}\cot\frac{p\pi}{2q} \quad (0 < p < 2q)$

959. $\int_0^\infty \frac{\sinh ax}{\mathrm{e}^{bx} + 1}\mathrm{d}x = \frac{\pi}{2b}\csc\frac{a\pi}{b} - \frac{1}{2a} \quad (b > 0)$

960. $\int_0^\infty \frac{\sinh ax}{\mathrm{e}^{bx} - 1}\mathrm{d}x = \frac{1}{2a} - \frac{\pi}{2b}\cot\frac{a\pi}{b} \quad (b > 0)$

Ⅱ.1.4.3 反双曲函数的积分

961. $\int_0^1 \operatorname{arsinh} x \mathrm{d}x = 1 - \sqrt{2} + \ln(1+\sqrt{2})$

962. $\int_1^\infty \operatorname{arcosh} x \mathrm{d}x = \infty$

963. $\int_0^1 \frac{\operatorname{arsinh} x}{x} \mathrm{d}x = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{1}{(2n+1)^2}$

964. $\int_1^\infty \frac{\operatorname{arcosh} x}{x} \mathrm{d}x = \mp \infty$

965. $\int_0^1 \frac{\operatorname{arsinh} x}{\sqrt{x^2+1}} \mathrm{d}x = \frac{1}{2}[\ln(1+\sqrt{2})]^2$

966. $\int_0^1 \frac{x \operatorname{arsinh} x}{\sqrt{x^2+1}} \mathrm{d}x = \sqrt{2}\ln(1+\sqrt{2}) - 1$

967. $\int_0^\infty \frac{\operatorname{arsinh} x}{x^2+1} \mathrm{d}x = 2G$

[这里，G 为卡塔兰常数(见附录)，以下同]

968. $\int_1^\infty \frac{\operatorname{arcosh} x}{x^2-1} \mathrm{d}x = \frac{\pi^2}{4}$

969. $\int_0^\infty \frac{\operatorname{arsinh} x}{x^{p+1}} \mathrm{d}x = \frac{1}{2p\sqrt{\pi}} \Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{1-p}{2}\right) \quad (0 < p < 1)$

[这里，$\Gamma(z)$ 为伽马函数(见附录)，以下同]

970. $\int_1^\infty \frac{\operatorname{arcosh} x}{x^{p+1}} \mathrm{d}x = \frac{\sqrt{\pi}}{2p} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} \quad (p > 0)$

971. $\int_1^\infty \frac{\operatorname{arcosh} x}{x^{2n}} \mathrm{d}x = \frac{(2n-1)!!}{2^n (2n-1)^2 (n-1)!} \quad (n = 1, 2, \cdots)$

972. $\int_1^\infty \frac{\operatorname{arcosh} x}{x^{2n+1}} \mathrm{d}x = \frac{2^{n-2}(n-1)!}{n(2n-1)!!} \quad (n = 1, 2, \cdots)$

973. $\int_0^a \operatorname{artanh} \frac{x}{a} \mathrm{d}x = a\ln 2$

974. $\int_0^a \left(\operatorname{artanh} \frac{x}{a}\right)^p \mathrm{d}x = 2^{1-p}\Gamma(p+1) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad (p > 0)$

975. $\int_0^a x \operatorname{artanh} \frac{x}{a} \mathrm{d}x = \frac{a^2}{2}$

976. $\int_0^a \operatorname{artanh} \frac{x}{a} \frac{\mathrm{d}x}{x} = \frac{\pi^2}{8}$

977. $\int_a^{\infty} \operatorname{arcoth}\frac{x}{a}\frac{\mathrm{d}x}{x} = \frac{\pi^2}{8} \quad (a>0)$

978. $\int_0^{a} \operatorname{artanh}\frac{x}{a}\frac{\mathrm{d}x}{x^2} = \infty$

979. $\int_a^{\infty} \operatorname{arcoth}\frac{x}{a}\frac{\mathrm{d}x}{x^2} = \frac{1}{a}\ln 2 \quad (a>0)$

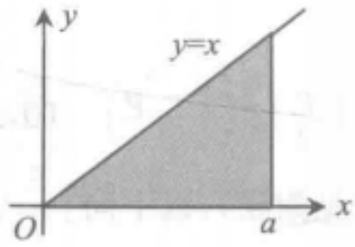
980. $\int_0^{a} \operatorname{artanh}\frac{x}{a}\frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = 2G$

981. $\int_a^{\infty} \operatorname{arcoth}\frac{x}{a}\frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \frac{\pi^2}{4} \quad (a>0)$

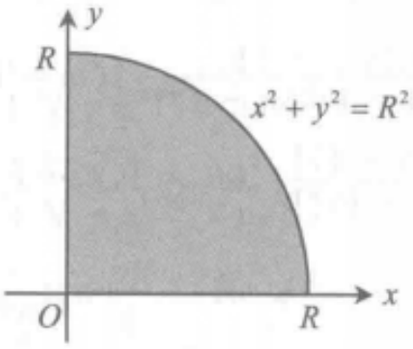
Ⅱ.1.5 重积分

Ⅱ.1.5.1 积分次序和积分变量交换的积分

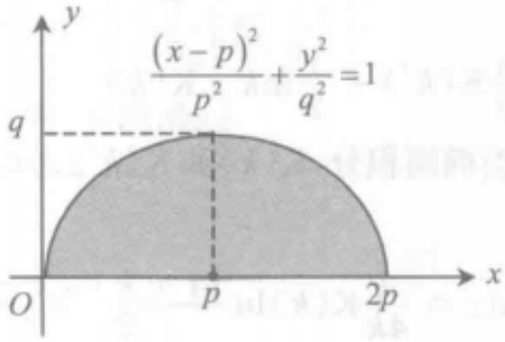
982. $\int_0^a \mathrm{d}x\int_0^x f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}y\int_y^a f(x,y)\mathrm{d}x$



983. $\int_0^R \mathrm{d}x\int_0^{\sqrt{R^2-x^2}} f(x,y)\mathrm{d}y = \int_0^R \mathrm{d}y\int_0^{\sqrt{R^2-y^2}} f(x,y)\mathrm{d}x$



984. $\int_0^{2p} \mathrm{d}x\int_0^{\frac{q}{p}\sqrt{2px-x^2}} f(x,y)\mathrm{d}y = \int_0^q \mathrm{d}y\int_{p\left[1-\sqrt{1-\left(\frac{y}{q}\right)^2}\right]}^{p\left[1+\sqrt{1-\left(\frac{y}{q}\right)^2}\right]} f(x,y)\mathrm{d}x$



Ⅱ.1.5.2 具有常数积分限的二重积分和三重积分

985. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\mathrm{d}x\mathrm{d}y = \frac{\pi}{2\sqrt{1-k^2}}$

986. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\cos y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\mathrm{d}x\mathrm{d}y = \mathrm{K}(k)$

[这里，$\mathrm{K}(k)$为第一类完全椭圆积分(见附录)，以下同]

987. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin\alpha\sin y}{\sqrt{1-\sin^2\alpha\sin^2 x\sin^2 y}}\mathrm{d}x\mathrm{d}y = \frac{\alpha\pi}{2}$

988. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{1-\cos x\cos y\cos z} = 4\pi\left[\mathrm{K}\left(\frac{\sqrt{2}}{2}\right)\right]^2$

989. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{3-\cos y\cos z-\cos x\cos z-\cos x\cos y} = \sqrt{3}\pi\left[\mathrm{K}\left(\sin\frac{\pi}{12}\right)\right]^2$

990. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{3-\cos x-\cos y-\cos z}$

$= 4\pi[18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}]\{\mathrm{K}[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})]\}^2$

Ⅱ.2 特殊函数的定积分

Ⅱ.2.1 椭圆函数的定积分

Ⅱ.2.1.1 椭圆积分的积分

这里，使用记号 $k'=\sqrt{1-k^2}$，并且 $k^2<1$，其中 k 称为模数，而 k' 称为

补模数.

1. $\int_0^{\frac{\pi}{2}} F(k,x)\cot x\,dx = \frac{\pi}{4}K(k') + \frac{1}{2}\ln k \cdot K(k)$

[这里,$F(k,x)$为第一类椭圆积分,$K(k)$和$K(k')$为第一类完全椭圆积分(见附录),以下同]

2. $\int_0^{\frac{\pi}{2}} F(k,x)\frac{\sin x\cos x}{1+k\sin^2 x}dx = \frac{1}{4k}K(k)\ln\frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k}K(k')$

3. $\int_0^{\frac{\pi}{2}} F(k,x)\frac{\sin x\cos x}{1-k\sin^2 x}dx = \frac{1}{4k}K(k)\ln\frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k}K(k')$

4. $\int_0^{\frac{\pi}{2}} F(k,x)\frac{\sin x\cos x}{1-k^2\sin^2 x}dx = -\frac{1}{2k^2}\ln k' \cdot K(k)$

5. $\int_0^{\frac{\pi}{2}} E(k,x)\frac{\sin x\cos x}{1-k^2\sin^2 x}dx = \frac{1}{2k^2}[(1+k'^2)K(k) - (2+\ln k')E(k)]$

[这里,$E(k,x)$为第二类椭圆积分,$E(k)$为第二类完全椭圆积分(见附录),以下同]

6. $\int_0^{\frac{\pi}{2}} E(k,x)\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2}[E(k)K(k) - \ln k']$

7. $\int_0^{\frac{\pi}{2}} E(k,\sin x)\frac{\sin x}{\sqrt{1-k^2\sin^2 x}}dx = \frac{\pi}{2k'}$

Ⅱ.2.1.2 椭圆积分相对于模数的积分

8. $\int_0^1 F(k,x)k\,dk = \frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$

9. $\int_0^1 E(k,x)k\,dk = \frac{\sin^2 x + 1 - \cos x}{3\sin x}$

10. $\int_0^1 \Pi(r^2,k,x)k\,dk = \tan\frac{x}{2} - r\ln\sqrt{\frac{1+r\sin x}{1-r\sin x}} - r^2\Pi(r^2,0,x)$

[这里,$\Pi(h,k,\varphi)$为第三类椭圆积分(见附录)]

Ⅱ.2.1.3 完全椭圆积分相对于模数的积分

11. $\int_0^1 K(k)dk = 2G$

[这里,G 为卡塔兰常数(见附录),以下同]

12. $\int_0^1 K(k')dk = \frac{\pi^2}{4}$

13. $\int_0^1 \frac{1}{k}\left[K(k) - \frac{\pi}{2}\right]dk = \pi\ln 2 - 2G$

14. $\int_0^1 \frac{K(k)}{k'}dk = \left[K\left(\frac{\sqrt{2}}{2}\right)\right]^2 = \frac{1}{16\pi}\left[\Gamma\left(\frac{1}{4}\right)\right]^4$

[这里，$\Gamma(z)$为伽马函数(见附录)]

15. $\int_0^1 \frac{K(k)}{1+k}dk = \frac{\pi^2}{8}$

16. $\int_0^1 E(k)dk = \frac{1}{2} + G$

17. $\int_0^1 E(k')dk = \frac{\pi^2}{8}$

18. $\int_0^1 \frac{1}{k}\left[E(k) - \frac{\pi}{2}\right]dk = \pi\ln 2 - 2G + 1 - \frac{\pi}{2}$

Ⅱ.2.2 指数积分、正弦积分等函数的定积分

Ⅱ.2.2.1 指数积分的积分

19. $\int_0^p Ei(ax)dx = pEi(ap) + \frac{1-e^{ap}}{a}$

[这里，$Ei(x)$为指数积分(见附录)，以下同]

20. $\int_0^\infty Ei(-px)x^{q-1}dx = -\frac{\Gamma(q)}{qp^q} \quad (\mathrm{Re}\ p \geqslant 0,\ \mathrm{Re}\ q > 0)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

21. $\int_0^\infty Ei(px)e^{-qx}dx = -\frac{1}{q}\ln\left(\frac{q}{p} - 1\right) \quad (p > 0,\ \mathrm{Re}\ q > 0,\ q > p)$

22. $\int_0^\infty Ei(-px)e^{-qx}dx = -\frac{1}{q}\ln\left(1 + \frac{q}{p}\right) \quad [\mathrm{Re}\ (p+q) \geqslant 0,\ q > 0]$

23. $\int_0^\infty Ei(-x)e^{-px}dx = \frac{1}{p(p+1)} - \frac{1}{p^2}\ln(1+p) \quad (\mathrm{Re}\ p > 0)$

24. $\int_0^\infty Ei(-x)e^x x^{\nu-1}dx = -\frac{\pi\Gamma(\nu)}{\sin\nu\pi} \quad (0 < \mathrm{Re}\ \nu < 1)$

25. $\int_0^\infty Ei(-ax)\sin bx dx = -\frac{1}{2b}\ln\left(1 + \frac{b^2}{a^2}\right) \quad (a > 0,\ b > 0)$

26. $\int_0^\infty Ei(-ax)\cos bx dx = -\frac{1}{b}\arctan\frac{b}{a} \quad (a > 0,\ b > 0)$

27. $\int_0^\infty Ei(-x)e^{-ax}\sin bx dx = -\frac{1}{a^2+b^2}\left\{\frac{b}{2}\ln[(1+a^2)+b^2] - a\arctan\frac{b}{1+a}\right\}$

$(\mathrm{Re}\ a > |\ \mathrm{Im}\ b\ |)$

28. $\int_0^\infty \mathrm{Ei}(-x)\mathrm{e}^{-ax}\cos bx\,\mathrm{d}x = -\frac{1}{a^2+b^2}\left\{\frac{a}{2}\ln[(1+a^2)+b^2]+b\arctan\frac{b}{1+a}\right\}$

(Re $a > |$ Im b $|$)

29. $\int_0^\infty \mathrm{Ei}(-x)\ln x\,\mathrm{d}x = 1+\gamma$

[这里，γ 为欧拉常数(见附录)]

Ⅱ.2.2.2 对数积分的积分

30. $\int_0^1 \mathrm{li}(x)\,\mathrm{d}x = -\ln 2$

[这里，$\mathrm{li}(x)$ 为对数积分(见附录)，以下同]

31. $\int_0^1 \mathrm{li}(x)x^{p-1}\,\mathrm{d}x = -\frac{1}{p}\ln(p+1) \quad (p>-1)$

32. $\int_0^1 \frac{\mathrm{li}(x)}{x^{q+1}}\,\mathrm{d}x = \frac{1}{q}\ln(1-q) \quad (q<1)$

33. $\int_1^\infty \frac{\mathrm{li}(x)}{x^{q+1}}\,\mathrm{d}x = -\frac{1}{q}\ln(q-1) \quad (q>1)$

Ⅱ.2.2.3 正弦积分和余弦积分函数的积分

34. $\int_0^p \mathrm{si}(\alpha x)\,\mathrm{d}x = p\,\mathrm{si}(p\alpha)+\frac{\cos p\alpha - 1}{\alpha}$

[这里，$\mathrm{si}(z)$ 为正弦积分(见附录)，以下同]

35. $\int_0^p \mathrm{ci}(\alpha x)\,\mathrm{d}x = p\,\mathrm{ci}(p\alpha)-\frac{\sin p\alpha}{\alpha}$

[这里，$\mathrm{ci}(z)$ 为余弦积分(见附录)，以下同]

36. $\int_0^\infty \frac{\mathrm{ci}(ax)}{b+x}\,\mathrm{d}x = -\frac{1}{2}\{[\mathrm{si}(ab)]^2+[\mathrm{ci}(ab)]^2\} \quad (a>0,\ |\arg b|<\pi)$

37. $\int_{-\infty}^\infty \frac{\mathrm{si}(a\,|\,x\,|)}{x-b}\mathrm{sgn}x\,\mathrm{d}x = \pi\,\mathrm{ci}(a\,|\,b\,|) \quad (a>0,\ b>0)$

38. $\int_{-\infty}^\infty \frac{\mathrm{ci}(a\,|\,x\,|)}{x-b}\,\mathrm{d}x = -\pi\cdot\mathrm{sgn}b\cdot\mathrm{si}(a\,|\,b\,|) \quad (a>0)$

39. $\int_0^\infty \mathrm{si}(px)\frac{x}{q^2+x^2}\,\mathrm{d}x = \frac{\pi}{2}\mathrm{Ei}(-pq) \quad (p>0,\ q>0)$

[这里，$\mathrm{Ei}(x)$ 为指数积分(见附录)，以下同]

40. $\int_0^\infty \mathrm{ci}(px)\frac{\mathrm{d}x}{q^2+x^2} = \frac{\pi}{2q}\mathrm{Ei}(-pq) \quad (p>0,\ q>0)$

41. $\int_0^\infty \mathrm{si}(px)\frac{x}{q^2-x^2}\,\mathrm{d}x = -\frac{\pi}{2}\mathrm{ci}(pq) \quad (p>0,\ q>0)$

42. $\int_0^\infty \mathrm{ci}(px)\frac{\mathrm{d}x}{q^2-x^2}=\frac{\pi}{2q}\mathrm{si}(pq)\quad(p>0,\ q>0)$

43. $\int_0^\infty \mathrm{si}(px)x^{q-1}\mathrm{d}x=-\frac{\Gamma(q)}{qp^q}\sin\frac{q\pi}{2}\quad(0<\mathrm{Re}\ q<1,\ p>0)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

44. $\int_0^\infty \mathrm{ci}(px)x^{q-1}\mathrm{d}x=-\frac{\Gamma(q)}{qp^q}\cos\frac{q\pi}{2}\quad(0<\mathrm{Re}\ q<1,\ p>0)$

45. $\int_0^\infty \mathrm{si}(px)\mathrm{e}^{-qx}\mathrm{d}x=-\frac{1}{q}\arctan\frac{q}{p}\quad(\mathrm{Re}\ q>0)$

46. $\int_0^\infty \mathrm{ci}(px)\mathrm{e}^{-qx}\mathrm{d}x=-\frac{1}{q}\ln\sqrt{1+\frac{q^2}{p^2}}\quad(\mathrm{Re}\ q>0)$

47. $\int_0^\infty \mathrm{si}(x)\mathrm{e}^{-qx^2}x\mathrm{d}x=\frac{\pi}{4q}\left[1-\Phi\left(\frac{1}{2\sqrt{q}}\right)\right]\quad(\mathrm{Re}\ q>0)$

[这里，$\Phi(x)$为概率积分(见附录)]

48. $\int_0^\infty \mathrm{ci}(x)\mathrm{e}^{-qx^2}x\mathrm{d}x=\frac{1}{4}\sqrt{\frac{\pi}{q}}\mathrm{Ei}\left(-\frac{1}{4q}\right)\quad(\mathrm{Re}\ q>0)$

49. $\int_0^\infty \sin px\ \mathrm{si}(qx)\mathrm{d}x=\begin{cases}-\frac{\pi}{2p} & (p^2>q^2)\\ -\frac{\pi}{4p} & (p^2=q^2)\\ 0 & (p^2<q^2)\end{cases}$

50. $\int_0^\infty \cos px\ \mathrm{si}(qx)\mathrm{d}x=\begin{cases}-\frac{1}{4p}\ln\left(\frac{p+q}{p-q}\right)^2 & (p\neq 0,\ p^2\neq q^2)\\ \frac{1}{q} & (p=0)\end{cases}$

51. $\int_0^\infty \sin px\ \mathrm{ci}(qx)\mathrm{d}x=\begin{cases}-\frac{1}{4p}\ln\left(\frac{p^2}{q^2}-1\right)^2 & (p\neq 0,\ p^2\neq q^2)\\ 0 & (p=0)\end{cases}$

52. $\int_0^\infty \cos px\ \mathrm{ci}(qx)\mathrm{d}x=\begin{cases}-\frac{\pi}{2p} & (p^2>q^2)\\ -\frac{\pi}{4p} & (p^2=q^2)\\ 0 & (p^2<q^2)\end{cases}$

53. $\int_0^\infty \mathrm{si}(x)\ln x\ \mathrm{d}x=1+\gamma$

[这里，γ 为欧拉常数(见附录)]

54. $\int_0^\infty \mathrm{ci}(x)\ln x\ \mathrm{d}x=\frac{\pi}{2}$

Ⅱ.2.2.4 概率积分函数的积分

55. $\int_0^p \Phi(ax)\mathrm{d}x = p\Phi(ap) + \frac{\mathrm{e}^{-a^2p^2}-1}{a\sqrt{\pi}}$

[这里,$\Phi(x)$为概率积分(见附录),以下同]

56. $\int_0^\infty \Phi(qx)\mathrm{e}^{-px}\mathrm{d}x = \frac{1}{p}\left[1-\Phi\left(\frac{p}{2q}\right)\right]\exp\left(\frac{p^2}{4q^2}\right) \quad \left(\mathrm{Re}\ p>0,\ |\arg q|<\frac{\pi}{4}\right)$

57. $\int_0^\infty \Phi(qx)\mathrm{e}^{-px^2}x\mathrm{d}x = \frac{q}{2p\sqrt{p+q^2}} \quad (\mathrm{Re}\ p>0,\ \mathrm{Re}\ p>-\mathrm{Re}\ q^2)$

58. $\int_0^\infty \Phi(qx)\mathrm{e}^{(q^2-p^2)x^2}x\mathrm{d}x = \frac{q}{2p(p^2-q^2)} \quad \left(\mathrm{Re}\ p^2>\mathrm{Re}\ q^2,\ |\arg p|<\frac{\pi}{4}\right)$

59. $\int_0^\infty \Phi(\mathrm{i}qx)\mathrm{e}^{-px^2}x\mathrm{d}x = \frac{\mathrm{i}q}{2p\sqrt{p-q^2}} \quad (q>0,\ \mathrm{Re}\ p>\mathrm{Re}\ q^2)$

60. $\int_0^\infty \Phi(\mathrm{i}x)\mathrm{e}^{-px-x^2}x\mathrm{d}x = \frac{\mathrm{i}}{\sqrt{\pi}}\left[\frac{1}{p}+\frac{p}{4}\mathrm{Ei}\left(-\frac{p^2}{4}\right)\right] \quad (\mathrm{Re}\ p>0)$

[这里,$\mathrm{Ei}(x)$为指数积分(见附录),以下同]

61. $\int_0^\infty \Phi(\mathrm{i}ax)\mathrm{e}^{-a^2x^2-bx}\mathrm{d}x = -\frac{1}{2a\mathrm{i}\sqrt{\pi}}\exp\left(\frac{b^2}{4a^2}\right)\mathrm{Ei}\left(-\frac{b^2}{4a^2}\right)$

$\left(\mathrm{Re}\ b>0,\ |\arg a|<\frac{\pi}{4}\right)$

62. $\int_0^\infty \Phi(x)\mathrm{e}^{-px^2}\frac{\mathrm{d}x}{x} = \frac{1}{2}\ln\frac{\sqrt{p+1}+1}{\sqrt{p+1}-1} = \mathrm{arcoth}\sqrt{p+1} \quad (\mathrm{Re}\ p>0)$

63. $\int_0^\infty \Phi(ax)\sin bx^2\mathrm{d}x = \frac{1}{4\sqrt{2\pi b}}\left(\ln\frac{a^2+a\sqrt{2b}+b}{a^2-a\sqrt{2b}+b}+2\arctan\frac{a\sqrt{2b}}{b-a^2}\right)$

$(a>0,\ b>0)$

Ⅱ.2.2.5 菲涅耳函数的积分

64. $\int_0^p \mathrm{S}(\alpha x)\mathrm{d}x = p\mathrm{S}(\alpha p) + \frac{\cos(\alpha^2p^2)-1}{\alpha\sqrt{2\pi}}$

[这里,$\mathrm{S}(z)$为菲涅耳函数(见附录),以下同]

65. $\int_0^p \mathrm{C}(\alpha x)\mathrm{d}x = p\mathrm{C}(\alpha p) - \frac{\sin(\alpha^2p^2)}{\alpha\sqrt{2\pi}}$

[这里,$\mathrm{C}(z)$为菲涅耳函数(见附录),以下同]

66. $\int_0^\infty \mathrm{S}(t)\mathrm{e}^{-pt}\mathrm{d}t = \frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-\mathrm{C}\left(\frac{p}{2}\right)\right]+\sin\frac{p^2}{4}\left[\frac{1}{2}-\mathrm{S}\left(\frac{p}{2}\right)\right]\right\}$

67. $\int_0^\infty C(t)e^{-pt}dt = \frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2} - S\left(\frac{p}{2}\right)\right] - \sin\frac{p^2}{4}\left[\frac{1}{2} - C\left(\frac{p}{2}\right)\right]\right\}$

68. $\int_0^\infty S(\sqrt{t})e^{-pt}dt = \frac{\sqrt{\sqrt{p^2+1}-p}}{2p\sqrt{p^2+1}}$

69. $\int_0^\infty C(\sqrt{t})e^{-pt}dt = \frac{\sqrt{\sqrt{p^2+1}+p}}{2p\sqrt{p^2+1}}$

70. $\int_0^\infty S(x)\sin b^2x^2dx = \begin{cases} 2^{-\frac{5}{2}}\frac{\sqrt{\pi}}{b} & (0 < b^2 < 1) \\ 0 & (b^2 > 1) \end{cases}$

71. $\int_0^\infty C(x)\cos b^2x^2dx = \begin{cases} 2^{-\frac{5}{2}}\frac{\sqrt{\pi}}{b} & (0 < b^2 < 1) \\ 0 & (b^2 > 1) \end{cases}$

Ⅱ.2.3 伽马(Gamma)函数的定积分

Ⅱ.2.3.1 伽马函数的积分

72. $\int_{-\infty}^\infty \Gamma(a+x)\Gamma(b-x)dx$

$$= \begin{cases} -i\pi 2^{1-a-b}\Gamma(a+b) & [\mathrm{Re}\,(a+b) < 1,\ \mathrm{Im}\,a > 0,\ \mathrm{Im}\,b > 0] \\ i\pi 2^{1-a-b}\Gamma(a+b) & [\mathrm{Re}\,(a+b) < 1,\ \mathrm{Im}\,a < 0,\ \mathrm{Im}\,b < 0] \\ 0 & [\mathrm{Re}\,(a+b) < 1,\ \mathrm{Im}\,a < 0,\ \mathrm{Im}\,b < 0] \end{cases}$$

[这里,$\Gamma(z)$为伽马函数(见附录),以下同]

73. $\int_0^\infty |\Gamma(a+ix)\Gamma(b+ix)|^2dx$

$$= \frac{\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{2\Gamma\left(a+b+\frac{1}{2}\right)} \quad (a > 0,\ b > 0)$$

74. $\int_0^\infty \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2 dx = \frac{\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b-a-\frac{1}{2}\right)}{2\Gamma(b)\Gamma\left(b-\frac{1}{2}\right)\Gamma(b-a)} \quad \left(0 < a < b - \frac{1}{2}\right)$

75. $\int_{-\infty}^\infty \frac{\Gamma(a+x)}{\Gamma(b+x)}dx = 0 \quad [\mathrm{Im}\,a \neq 0,\ \mathrm{Re}\,(a-b) < -1]$

76. $\int_{-\infty}^\infty \frac{dx}{\Gamma(a+x)\Gamma(b-x)} = \frac{2^{a+b-2}}{\Gamma(a+b-1)} \quad [\mathrm{Re}\,(a+b) > 1]$

Ⅱ.2.3.2 伽马函数与三角函数组合的积分

77. $$\int_{-\infty}^{\infty}\frac{\sin rx}{\Gamma(p+x)\Gamma(q-x)}\mathrm{d}x=\begin{cases}\dfrac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\sin\frac{r(q-p)}{2}}{\Gamma(p+q-1)} & (|r|<\pi)\\ 0 & (|r|>\pi)\end{cases}$$

[Re $(p+q)>1$，r 为实数]

78. $$\int_{-\infty}^{\infty}\frac{\cos rx}{\Gamma(p+x)\Gamma(q-x)}\mathrm{d}x=\begin{cases}\dfrac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\cos\frac{r(q-p)}{2}}{\Gamma(p+q-1)} & (|r|<\pi)\\ 0 & (|r|>\pi)\end{cases}$$

[Re $(p+q)>1$，r 为实数]

79. $$\int_{-\infty}^{\infty}\frac{\sin m\pi x}{\sin\pi x\Gamma(p+x)\Gamma(q-x)}\mathrm{d}x=\begin{cases}\dfrac{2^{p+q-2}}{\Gamma(p+q-1)} & (m\text{ 为奇数})\\ 0 & (m\text{ 为偶数})\end{cases}$$

Ⅱ.2.3.3 伽马函数的对数的积分

80. $$\int_{p}^{p+1}\ln\Gamma(x)\mathrm{d}x=\frac{1}{2}\ln 2\pi+p\ln p-p$$

81. $$\int_{0}^{1}\ln\Gamma(x)\mathrm{d}x=\int_{0}^{1}\ln\Gamma(1-x)\mathrm{d}x=\frac{1}{2}\ln 2\pi$$

82. $$\int_{0}^{1}\ln\Gamma(x+q)\mathrm{d}x=\frac{1}{2}\ln 2\pi+q\ln q-q\quad(q\geqslant 0)$$

83. $$\int_{0}^{z}\ln\Gamma(x+1)\mathrm{d}x=\frac{z}{2}\ln 2\pi-\frac{z(z+1)}{2}+z\ln\Gamma(z+1)-\ln G(z+1)$$

{这里，

$$G(z+1)=(2\pi)^{\frac{z}{2}}\exp\left[-\frac{z(z+1)}{2}-\frac{z^2\gamma}{2}\right]\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^{k}\exp\left(-z+\frac{z^2}{2k}\right)\right]$$

其中，γ 为欧拉常数}

84. $$\int_{0}^{n}\ln\Gamma(a+x)\mathrm{d}x=\sum_{k=0}^{n-1}(a+k)\ln(a+k)-na+\frac{1}{2}n\ln(2\pi)-\frac{1}{2}n(n-1)$$

$(a\geqslant 0,\ n=1,2,\cdots)$

85. $$\int_{0}^{1}\ln\Gamma(a+x)\exp(2n\pi x\mathrm{i})\mathrm{d}x=\frac{1}{2n\pi\mathrm{i}}[\ln a-\exp(-2n\pi a\mathrm{i})\mathrm{Ei}(2n\pi a\mathrm{i})]$$

$(a>0,\ n=\pm 1,\pm 2,\cdots)$

[这里，$\mathrm{Ei}(x)$ 为指数积分(见附录)]

86. $\int_0^1 \ln\Gamma(x)\sin 2n\pi x\,\mathrm{d}x = \frac{1}{2n\pi}[\ln(2n\pi)+\gamma]$

[这里，γ 为欧拉常数(见附录)，以下同]

87. $\int_0^1 \ln\Gamma(x)\cos 2n\pi x\,\mathrm{d}x = \frac{1}{4n}$

88. $\int_0^1 \ln\Gamma(x)\sin(2n+1)\pi x\,\mathrm{d}x$

$$= \frac{1}{(2n+1)\pi}\left[\ln\frac{\pi}{2} + 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)+\frac{1}{2n+1}\right]$$

89. $\int_0^1 \ln\Gamma(x)\cos(2n+1)\pi x\,\mathrm{d}x = \frac{2}{\pi^2}\left[\frac{\ln(2\pi)+\gamma}{(2n+1)^2} + 2\sum_{k=2}^{\infty}\frac{\ln k}{4k^2-(2n+1)^2}\right]$

90. $\int_0^1 \ln\Gamma(a+x)\sin 2n\pi x\,\mathrm{d}x$

$$= -\frac{1}{2n\pi}[\ln a + \cos 2n\pi a\,\mathrm{ci}(2n\pi a) - \sin 2n\pi a\,\mathrm{si}(2n\pi a)] \quad (a>0,\ n=1,2,\cdots)$$

[这里，si(x)和 ci(x)分别为正弦积分和余弦积分(见附录)，以下同]

91. $\int_0^1 \ln\Gamma(a+x)\cos 2n\pi x\,\mathrm{d}x = -\frac{1}{2n\pi}[\sin 2n\pi a\,\mathrm{ci}(2n\pi a) + \cos 2n\pi a\,\mathrm{si}(2n\pi a)]$

$(a>0,\ n=1,2,\cdots)$

Ⅱ.2.3.4 ψ 函数的积分

92. $\int_1^x \psi(x)\mathrm{d}x = \ln\Gamma(x)$

[这里，Psi 函数 $\psi(x)$ 的定义见附录，以下同]

93. $\int_0^1 \psi(a+x)\mathrm{d}x = \ln a \quad (a>0)$

94. $\int_0^1 \mathrm{e}^{2\pi n x\mathrm{i}}\psi(a+x)\mathrm{d}x = \mathrm{e}^{-2\pi n a\mathrm{i}}\mathrm{Ei}(2\pi n a\mathrm{i}) \quad (a>0,\ n=\pm 1,\pm 2,\cdots)$

[这里，Ei(x)为指数积分(见附录)]

95. $\int_0^1 \psi(x)\sin\pi x\,\mathrm{d}x = -\frac{2}{\pi}\left[\gamma+\ln(2\pi)+2\sum_{k=2}^{\infty}\frac{\ln k}{4k^2-1}\right]$

[这里，γ 为欧拉常数(见附录)，以下同]

96. $\int_0^1 \psi(x)\sin^2\pi x\,\mathrm{d}x = -\frac{1}{2}[\gamma+\ln(2\pi)]$

97. $\int_0^1 \psi(x)\sin\pi x\cos\pi x\,\mathrm{d}x = -\frac{\pi}{4}$

98. $\int_0^1 \psi(x)\sin\pi x\sin n\pi x\,\mathrm{d}x = \begin{cases}\frac{1}{2}\ln\frac{n-1}{n+1} & (n\text{ 为奇数})\\ \frac{n}{1-n^2} & (n\text{ 为偶数})\end{cases}$

99. $\int_0^1 \psi(x)\sin 2n\pi x\,\mathrm{d}x = -\frac{1}{2}\pi \quad (n = 1,2,\cdots)$

100. $\int_0^1 [\psi(a+\mathrm{i}x) - \psi(a-\mathrm{i}x)]\sin xy\mathrm{d}x = \mathrm{i}\pi\frac{\mathrm{e}^{-ay}}{1-\mathrm{e}^{-y}} \quad (a>0,\ y>0)$

101. $\int_0^1 \psi(a+x)\sin 2n\pi x\,\mathrm{d}x = \sin 2n\pi a\,\mathrm{ci}(2n\pi a) + \cos 2n\pi a\,\mathrm{si}(2n\pi a)$

$(a \geqslant 0,\ n = 1,2,\cdots)$

[这里，$\mathrm{si}(x)$和$\mathrm{ci}(x)$分别为正弦积分和余弦积分(见附录)，以下同]

102. $\int_0^1 \psi(a+x)\cos 2n\pi x\,\mathrm{d}x = \sin 2n\pi a\,\mathrm{si}(2n\pi a) - \cos 2n\pi a\,\mathrm{ci}(2n\pi a)$

$(a>0,\ n = 1,2,\cdots)$

103. $\int_0^\infty [\psi(1+x) - \ln x]\cos(2\pi xy)\mathrm{d}x = \frac{1}{2}[\psi(y+1) - \ln y]$

Ⅱ.2.4 贝塞尔(Bessel)函数的定积分

Ⅱ.2.4.1 贝塞尔函数的积分

104. $\int_0^\infty \mathrm{J}_\nu(bx)\mathrm{d}x = \frac{1}{b} \quad (\mathrm{Re}\,\nu > -1,\ b>0)$

[这里，$\mathrm{J}_\nu(z)$为贝塞尔函数(见附录)，以下同]

105. $\int_0^\infty \mathrm{N}_\nu(bx)\mathrm{d}x = -\frac{1}{b}\tan\frac{\nu\pi}{2} \quad (|\mathrm{Re}\,\nu| < 1,\ b>0)$

[这里，$\mathrm{N}_\nu(z)$为贝塞尔函数(见附录)，以下同]

106. $\int_0^a \mathrm{J}_\nu(x)\mathrm{d}x = 2\sum_{k=0}^\infty \mathrm{J}_{\nu+2k+1}(a) \quad (\mathrm{Re}\,\nu > -1)$

107. $\int_0^a \mathrm{J}_{\frac{1}{2}}(x)\mathrm{d}x = 2\mathrm{S}(\sqrt{a})$

[这里，$\mathrm{S}(z)$为菲涅耳函数(见附录)]

108. $\int_0^a \mathrm{J}_{-\frac{1}{2}}(x)\mathrm{d}x = 2\mathrm{C}(\sqrt{a})$

[这里，$\mathrm{C}(z)$为菲涅耳函数(见附录)]

109. $\int_0^a \mathrm{J}_1(x)\mathrm{d}x = 1 - \mathrm{J}_0(a) \quad (a>0)$

110. $\int_a^\infty \mathrm{J}_1(x)\mathrm{d}x = \mathrm{J}_0(a) \quad (a>0)$

111. $\int_0^z \mathrm{J}_0(\sqrt{z^2 - x^2})\mathrm{d}x = \sin z$

112. $\int_0^{\frac{\pi}{2}} J_{2\nu}(2z\sin x)\mathrm{d}x = \frac{\pi}{2}[J_\nu(z)]^2 \quad \left(\mathrm{Re}\,\nu > -\frac{1}{2}\right)$

113. $\int_0^{\frac{\pi}{2}} J_{2\nu}(2z\cos x)\mathrm{d}x = \frac{\pi}{2}[J_\nu(z)]^2 \quad \left(\mathrm{Re}\,\nu > -\frac{1}{2}\right)$

114. $\int_0^{\infty} K_{2\nu}(2z\sinh x)\mathrm{d}x = \frac{\pi^2}{8\cos\nu\pi}\{[J_\nu(z)]^2 + [N_\nu(z)]^2\}$

$\left(\mathrm{Re}\,z > 0,\ -\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}\right)$

Ⅱ.2.4.2 贝塞尔函数与 x 组合的积分

115. $\int_0^1 xJ_\nu(ax)J_\nu(bx)\mathrm{d}x = \begin{cases} \frac{1}{2}[J_{\nu+1}(a)]^2 & (a = b) \\ 0 & (a \neq b) \end{cases}$

$[J_\nu(a) = J_\nu(b) = 0,\ \nu > -1]$

116. $\int_0^1 xJ_\nu(ax)J_\nu(bx)\mathrm{d}x = \begin{cases} \frac{1}{2}\left[1 - \left(\frac{\nu}{a}\right)^2\right][J_\nu(a)]^2 & (a = b) \\ 0 & (a \neq b) \end{cases}$

$[J_\nu'(a) = J_\nu'(b) = 0,\ \nu > -1]$

117. $\int_0^1 xJ_\nu(ax)J_\nu(bx)\mathrm{d}x = \begin{cases} \frac{1}{2}\left[1 - \left(\frac{\nu}{a}\right)^2 + \left(\frac{h}{a}\right)^2\right][J_\nu(a)]^2 & (a = b) \\ 0 & (a \neq b) \end{cases}$

$[aJ_\nu'(a) + hJ_\nu(a) = bJ_\nu'(b) + hJ_\nu(b) = 0,\ a > 0,\ b > 0,\ h > 0,\ \nu > -1]$

118. $\int_0^{\infty} xK_\nu(ax)K_\nu(bx)\mathrm{d}x = \frac{\pi(a^{2\nu} - b^{2\nu})}{2(a^2 - b^2)(ab)^\nu \sin\nu\pi}$

$[|\,\mathrm{Re}\,\nu\,| < 1,\ \mathrm{Re}\,(a + b) > 0]$

Ⅱ.2.4.3 贝塞尔函数与代数函数组合的积分

119. $\int_0^{\infty} \frac{xJ_0(bx)}{x^2 + a^2}\mathrm{d}x = K_0(ab) \quad (b > 0,\ \mathrm{Re}\,a > 0)$

[这里，$K_\nu(z)$ 为第二类虚自变量贝塞尔函数(见附录)，以下同]

120. $\int_0^{\infty} \frac{N_0(bx)}{x^2 + a^2}\mathrm{d}x = -\frac{K_0(ab)}{a} \quad (b > 0,\ \mathrm{Re}\,a > 0)$

121. $\int_0^{\infty} \frac{J_\nu(xy)}{\sqrt{x^2 + a^2}}\mathrm{d}x = I_{\frac{\nu}{2}}\left(\frac{ay}{2}\right)K_{\frac{\nu}{2}}\left(\frac{ay}{2}\right) \quad (\mathrm{Re}\,a > 0,\ y > 0,\ \mathrm{Re}\,\nu > -1)$

[这里，$I_\nu(z)$ 为第一类虚自变量贝塞尔函数(见附录)，以下同]

122. $\int_0^{\infty} \frac{N_\nu(xy)}{\sqrt{x^2 + a^2}}\mathrm{d}x = -\frac{1}{\pi}\sec\frac{\nu\pi}{2}K_{\frac{\nu}{2}}\left(\frac{ay}{2}\right)\left[K_{\frac{\nu}{2}}\left(\frac{ay}{2}\right) + \pi\sin\frac{\nu\pi}{2}I_{\frac{\nu}{2}}\left(\frac{ay}{2}\right)\right]$

$(\mathrm{Re}\, a > 0,\ y > 0,\ |\mathrm{Re}\, \nu| < 1)$

123. $\int_0^\infty \frac{K_\nu(xy)}{\sqrt{x^2+a^2}}dx = \frac{\pi^2}{8}\sec\frac{\nu\pi}{2}\left\{\left[J_{\frac{\nu}{2}}\left(\frac{ay}{2}\right)\right]^2 + \left[N_{\frac{\nu}{2}}\left(\frac{ay}{2}\right)\right]^2\right\}$

$(\mathrm{Re}\, a > 0,\ \mathrm{Re}\, y > 0,\ |\mathrm{Re}\, \nu| < 1)$

124. $\int_0^1 \frac{J_\nu(xy)}{\sqrt{1-x^2}}dx = \frac{\pi}{2}\left[J_{\frac{\nu}{2}}\left(\frac{y}{2}\right)\right]^2 \quad (y > 0,\ \mathrm{Re}\, \nu > -1)$

125. $\int_0^1 \frac{N_0(xy)}{\sqrt{1-x^2}}dx = \frac{\pi}{2}J_0\left(\frac{y}{2}\right)N_0\left(\frac{y}{2}\right) \quad (y > 0)$

126. $\int_1^\infty \frac{J_\nu(xy)}{\sqrt{x^2-1}}dx = -\frac{\pi}{2}J_{\frac{\nu}{2}}\left(\frac{y}{2}\right)N_{\frac{\nu}{2}}\left(\frac{y}{2}\right) \quad (y > 0)$

127. $\int_1^\infty \frac{N_\nu(xy)}{\sqrt{x^2-1}}dx = \frac{\pi}{4}\left\{\left[J_{\frac{\nu}{2}}\left(\frac{y}{2}\right)\right]^2 - \left[N_{\frac{\nu}{2}}\left(\frac{y}{2}\right)\right]^2\right\} \quad (y > 0)$

128. $\int_0^\infty \frac{xJ_0(xy)}{\sqrt{a^2+x^2}}dx = \frac{1}{y}e^{-ay} \quad (y > 0,\ \mathrm{Re}\, a > 0)$

129. $\int_0^1 \frac{xJ_0(xy)}{\sqrt{1-x^2}}dx = \frac{1}{y}\sin y \quad (y > 0)$

130. $\int_1^\infty \frac{xJ_0(xy)}{\sqrt{x^2-1}}dx = \frac{1}{y}\cos y \quad (y > 0)$

131. $\int_0^\infty \frac{xJ_0(xy)}{\sqrt{(x^2+a^2)^3}}dx = \frac{1}{a}e^{-ay} \quad (y > 0,\ \mathrm{Re}\, a > 0)$

132. $\int_0^\infty \frac{xJ_0(ax)}{\sqrt{x^4+4b^4}}dx = K_0(ab)J_0(ab) \quad (a > 0,\ b > 0)$

133. $\int_0^\infty \frac{J_\nu(a\sqrt{x^2+1})}{\sqrt{x^2+1}}dx = -\frac{\pi}{2}J_{\frac{\nu}{2}}\left(\frac{a}{2}\right)N_{\frac{\nu}{2}}\left(\frac{a}{2}\right) \quad (a > 0,\ \mathrm{Re}\, \nu > -1)$

Ⅱ.2.4.4 贝塞尔函数与幂函数组合的积分

134. $\int_0^1 x^{\nu+1}J_\nu(ax)dx = a^{-1}J_{\nu+1}(a) \quad \left(\mathrm{Re}\, \nu > -1\right)$

135. $\int_0^1 x^{\nu+1}N_\nu(ax)dx = a^{-1}N_{\nu+1}(a) + 2^{\nu+1}a^{-\nu-2}\Gamma(\nu+1) \quad \left(\mathrm{Re}\, \nu > -1\right)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

136. $\int_0^1 x^{\nu+1}I_\nu(ax)dx = a^{-1}I_{\nu+1}(a) \quad \left(\mathrm{Re}\, \nu > -1\right)$

137. $\int_0^1 x^{\nu+1}K_\nu(ax)dx = -a^{-1}K_{\nu+1}(a) + 2^\nu a^{-\nu-2}\Gamma(\nu+1) \quad \left(\mathrm{Re}\, \nu > -1\right)$

138. $\int_0^1 x^{1-\nu}J_\nu(ax)dx = \frac{a^{\nu-2}}{2^{\nu-1}\Gamma(\nu)} - a^{-1}J_{\nu-1}(a)$

139. $\int_0^1 x^{1-\nu} N_\nu(ax)\mathrm{d}x = \frac{a^{\nu-2}\cot\nu\pi}{2^{\nu-1}\Gamma(\nu)} - a^{-1}N_{\nu-1}(a) \quad \left(\mathrm{Re}\,\nu < 1\right)$

140. $\int_0^1 x^{1-\nu} I_\nu(ax)\mathrm{d}x = a^{-1}I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1}\Gamma(\nu)}$

141. $\int_0^1 x^{1-\nu} K_\nu(ax)\mathrm{d}x = -a^{-1}K_{\nu-1}(a) + 2^{-\nu}a^{\nu-2}\Gamma(1-\nu) \quad \left(\mathrm{Re}\,\nu < 1\right)$

142. $\int_0^\infty x^\mu J_\nu(ax)\mathrm{d}x = 2^\mu a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)}$

$\left(a > 0,\ -\mathrm{Re}\,\nu - 1 < \mathrm{Re}\,\mu < \frac{1}{2}\right)$

143. $\int_0^\infty x^\mu N_\nu(ax)\mathrm{d}x = 2^\mu \cot\frac{(1+\nu-\mu)\pi}{2}\cdot a^{-\mu-1}\cdot\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)}$

$\left(a > 0,\ |\mathrm{Re}\,\nu| - 1 < \mu < \frac{1}{2}\right)$

144. $\int_0^\infty x^\mu K_\nu(ax)\mathrm{d}x = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)$

$[\mathrm{Re}\,a > 0,\ \mathrm{Re}\,(\mu+1\pm\nu) > 0]$

145. $\int_0^\infty \frac{J_\nu(ax)}{x^{\nu-\mu}}\mathrm{d}x = \frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{2}\right)}{2^{\nu-\mu}a^{\mu-\nu+1}\Gamma\left(\nu-\frac{\mu}{2}+\frac{1}{2}\right)} \quad \left(-1 < \mathrm{Re}\,\mu < \mathrm{Re}\,\nu - \frac{1}{2}\right)$

146. $\int_0^\infty \frac{N_\nu(ax)}{x^{\nu-\mu}}\mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}+\frac{\mu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\mu}{2}-\nu\right)\sin\left(\frac{\mu}{2}-\nu\right)\pi}{2^{\nu-\mu}\pi}$

$\left[|\mathrm{Re}\,\nu| < \mathrm{Re}\,(1+\mu-\nu) < \frac{3}{2}\right]$

Ⅱ.2.4.5 贝塞尔函数与三角函数组合的积分

147. $$\int_0^\infty J_\nu(ax)\sin bx\mathrm{d}x = \begin{cases} \dfrac{\sin\left(\nu\arcsin\frac{b}{a}\right)}{\sqrt{a^2-b^2}} & (b < a,\ \mathrm{Re}\,\nu > -2) \\ \infty \text{ 或 } 0 & (b = a,\ \mathrm{Re}\,\nu > -2) \\ \dfrac{a^\nu\cos\frac{\nu\pi}{2}}{\sqrt{b^2-a^2}\left(b+\sqrt{b^2-a^2}\right)^\nu} & (b > a,\ \mathrm{Re}\,\nu > -2) \end{cases}$$

148. $$\int_0^\infty J_\nu(ax)\cos bx\,dx = \begin{cases} \dfrac{\cos\left(\nu \arcsin \dfrac{b}{a}\right)}{\sqrt{a^2-b^2}} & (b<a,\ \mathrm{Re}\,\nu>-1) \\ \infty \text{ 或 } 0 & (b=a,\ \mathrm{Re}\,\nu>-1) \\ -\dfrac{a^\nu \sin\dfrac{\nu\pi}{2}}{\sqrt{b^2-a^2}\left(b+\sqrt{b^2-a^2}\right)^\nu} & (b>a,\ \mathrm{Re}\,\nu>-1) \end{cases}$$

149. $$\int_0^\infty N_\nu(ax)\sin bx\,dx$$

$$= \begin{cases} \dfrac{\cot\dfrac{\nu\pi}{2}}{\sqrt{a^2-b^2}}\sin\left(\nu \arcsin\dfrac{b}{a}\right) & (0<b<a,\ |\mathrm{Re}\,\nu|<2) \\ \dfrac{\csc\dfrac{\nu\pi}{2}}{2\sqrt{b^2-a^2}}\left[a^{-\nu}\cos\nu\pi\left(b-\sqrt{b^2-a^2}\right)^\nu - a^\nu\left(b-\sqrt{b^2-a^2}\right)^{-\nu}\right] & \\ & (0<a<b,\ |\mathrm{Re}\,\nu|<2) \end{cases}$$

150. $$\int_0^\infty N_\nu(ax)\cos bx\,dx$$

$$= \begin{cases} \dfrac{\tan\dfrac{\nu\pi}{2}}{\sqrt{a^2-b^2}}\cos\left(\nu \arcsin\dfrac{b}{a}\right) & (0<b<a,\ |\mathrm{Re}\,\nu|<1) \\ -\dfrac{\sin\dfrac{\nu\pi}{2}}{\sqrt{b^2-a^2}}\left[a^{-\nu}\left(b-\sqrt{b^2-a^2}\right)^\nu + \cot\nu\pi\right. & \\ \left. \quad + a^\nu\csc\nu\pi\left(b-\sqrt{b^2-a^2}\right)^{-\nu}\right] & (0<a<b,\ |\mathrm{Re}\,\nu|<1) \end{cases}$$

151. $$\int_0^\infty K_\nu(ax)\sin bx\,dx = \frac{\pi a^{-\nu}\csc\dfrac{\nu\pi}{2}}{4\sqrt{a^2+b^2}}\left[\left(\sqrt{b^2+a^2}+b\right)^\nu - \left(\sqrt{b^2+a^2}-b\right)^\nu\right]$$

$(\mathrm{Re}\,a>0,\ b>0,\ |\mathrm{Re}\,\nu|<2,\ \nu\neq 0)$

152. $$\int_0^\infty K_\nu(ax)\cos bx\,dx$$

$$= \frac{\pi\sec\dfrac{\nu\pi}{2}}{4\sqrt{a^2+b^2}}\left[a^{-\nu}\left(\sqrt{b^2+a^2}+b\right)^\nu + a^\nu\left(\sqrt{b^2+a^2}+b\right)^{-\nu}\right]$$

$(\mathrm{Re}\,a>0,\ b>0,\ |\mathrm{Re}\,\nu|<1)$

153. $$\int_0^\infty J_0(ax)\sin bx\,dx = \begin{cases} 0 & (0<b<a) \\ \dfrac{1}{\sqrt{b^2-a^2}} & (0<a<b) \end{cases}$$

154. $\int_0^\infty J_0(ax)\cos bx\,dx = \begin{cases} \dfrac{1}{\sqrt{a^2-b^2}} & (0<b<a) \\ \infty & (a=b) \\ 0 & (0<a<b) \end{cases}$

155. $\int_0^\infty N_0(ax)\sin bx\,dx = \begin{cases} \dfrac{2}{\pi}\dfrac{1}{\sqrt{a^2-b^2}}\arcsin\dfrac{b}{a} & (0<b<a) \\ \dfrac{2}{\pi}\dfrac{1}{\sqrt{b^2-a^2}}\ln\left(\dfrac{b}{a}-\sqrt{\dfrac{b^2}{a^2}-1}\right) & (0<a<b) \end{cases}$

156. $\int_0^\infty N_0(ax)\cos bx\,dx = \begin{cases} 0 & (0<b<a) \\ -\dfrac{1}{\sqrt{b^2-a^2}} & (0<a<b) \end{cases}$

157. $\int_0^\infty K_0(ax)\sin bx\,dx = \dfrac{1}{\sqrt{b^2+a^2}}\ln\left(\dfrac{b}{a}+\sqrt{\dfrac{b^2}{a^2}+1}\right) \quad (a>0,\ b>0)$

158. $\int_0^\infty K_0(ax)\cos bx\,dx = \dfrac{\pi}{2\sqrt{b^2+a^2}}$ （$a>0$，a 和 b 均为实数）

Ⅱ.2.4.6 贝塞尔函数与指数函数和幂函数组合的积分

159. $\int_0^\infty e^{-ax}J_\nu(bx)\,dx = \dfrac{b^{-\nu}\left(\sqrt{a^2+b^2}-a\right)^\nu}{\sqrt{a^2+b^2}} \quad [\mathrm{Re}\,\nu>-1,\ \mathrm{Re}\,(a\pm ib)>0]$

160. $\int_0^\infty e^{-ax}N_\nu(bx)\,dx$

$$= \frac{\csc\nu\pi}{\sqrt{a^2+b^2}}\left[b^\nu\left(\sqrt{a^2+b^2}+a\right)^{-\nu}\cos\nu\pi - b^{-\nu}\left(\sqrt{a^2+b^2}+a\right)^\nu\right]$$

$(\mathrm{Re}\,\nu>0,\ b>0,\ |\mathrm{Re}\,\nu|<1)$

161. $\int_0^\infty e^{-ax}H_\nu^{(1,2)}(bx)\,dx$

$$= \frac{\left(\sqrt{a^2+b^2}-a\right)^\nu}{b^\nu\sqrt{a^2+b^2}}\left\{1\pm\frac{i}{\sin\nu\pi}\left[\cos\nu\pi-\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)^{2\nu}\right]\right\}$$

$\left(-1<\mathrm{Re}\,\nu<1\text{；正号相应于函数 }H_\nu^{(1)}\text{，负号相应于函数 }H_\nu^{(2)}\right)$

162. $\int_0^\infty e^{-ax}I_\nu(bx)\,dx = \dfrac{(a-\sqrt{a^2-b^2})^\nu}{b^\nu\sqrt{a^2-b^2}} \quad \left(\mathrm{Re}\,a>|\mathrm{Re}\,b|,\ \mathrm{Re}\,\nu>-1\right)$

163. $\int_0^\infty e^{-ax}K_\nu(bx)\,dx$

$$= \begin{cases} \dfrac{\pi}{b\sin\nu\pi}\cdot\dfrac{\sin\nu\theta}{\sin\theta} & \left(\cos\theta = \dfrac{a}{b}, \text{当 } b\to\infty \text{ 时}, \theta\to\dfrac{\pi}{2}\right) \\ \dfrac{\pi\csc\nu\pi}{2\sqrt{a^2-b^2}}\left[b^{-\nu}\left(a+\sqrt{a^2-b^2}\right)^{\nu} - b^{\nu}\left(a+\sqrt{a^2-b^2}\right)^{-\nu}\right] & \\ \qquad\left(\mathrm{Re}\,(a+b)>0,\ |\,\mathrm{Re}\,\nu\,|<1\right) & \end{cases}$$

164. $\displaystyle\int_0^\infty e^{-ax}N_0(bx)dx = -\frac{2}{\pi\sqrt{a^2+b^2}}\ln\frac{a+\sqrt{a^2+b^2}}{b}$ $\left(\mathrm{Re}\,a>|\,\mathrm{Im}\,b\,|\right)$

165. $\displaystyle\int_0^\infty e^{-ax}H_0^{(1)}(bx)dx = \frac{1}{\sqrt{a^2+b^2}}\left\{1-\frac{2i}{\pi}\ln\left[\frac{a}{b}+\sqrt{1+\left(\frac{a}{b}\right)^2}\right]\right\}$

$\left(\mathrm{Re}\,a>|\,\mathrm{Im}\,b\,|\right)$

166. $\displaystyle\int_0^\infty e^{-ax}H_0^{(2)}(bx)dx = \frac{1}{\sqrt{a^2+b^2}}\left\{1+\frac{2i}{\pi}\ln\left[\frac{a}{b}+\sqrt{1+\left(\frac{a}{b}\right)^2}\right]\right\}$

$\left(\mathrm{Re}\,a>|\,\mathrm{Im}\,b\,|\right)$

167. $\displaystyle\int_0^\infty e^{-ax}K_0(bx)dx$

$$= \begin{cases} \dfrac{1}{\sqrt{b^2-a^2}}\arccos\dfrac{a}{b} & [0<a<b,\ \mathrm{Re}\,(a+b)>0] \\ \dfrac{1}{\sqrt{a^2-b^2}}\ln\left(\dfrac{a}{b}+\sqrt{\dfrac{a^2}{b^2}-1}\right) & [0\leqslant b<a,\ \mathrm{Re}\,(a+b)>0] \end{cases}$$

168. $\displaystyle\int_0^\infty e^{-ax^2}J_\nu(bx)dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\exp\left(-\frac{b^2}{8a}\right)I_{\frac{\nu}{2}}\left(\frac{b^2}{8a}\right)$

$(\mathrm{Re}\,a>0,\ b>0,\ \mathrm{Re}\,\nu>-1)$

169. $\displaystyle\int_0^\infty e^{-ax^2}N_\nu(bx)dx$

$$= -\frac{\sqrt{\pi}}{2\sqrt{a}}\exp\left(-\frac{b^2}{8a}\right)\left[\tan\frac{\nu\pi}{2}I_{\frac{\nu}{2}}\left(\frac{b^2}{8a}\right)+\frac{1}{\pi}\sec\frac{\nu\pi}{2}K_{\frac{\nu}{2}}\left(\frac{b^2}{8a}\right)\right]$$

$(\mathrm{Re}\,a>0,\ b>0,\ |\,\mathrm{Re}\,\nu\,|<1)$

170. $\displaystyle\int_0^\infty e^{-ax^2}K_\nu(bx)dx = \frac{\sqrt{\pi}}{4\sqrt{a}}\sec\frac{\nu\pi}{2}\exp\left(\frac{b^2}{8a}\right)K_{\frac{\nu}{2}}\left(\frac{b^2}{8a}\right)$

$(\mathrm{Re}\,a>0,\ |\,\mathrm{Re}\,\nu\,|<1)$

171. $\displaystyle\int_0^\infty x^{m+1}e^{-ax}J_\nu(bx)dx = (-1)^{m+1}b^{-\nu}\frac{d^{m+1}}{da^{m+1}}\left[\frac{\left(\sqrt{a^2+b^2}-a\right)^{\nu}}{\sqrt{a^2+b^2}}\right]$

$(b>0,\ \mathrm{Re}\,\nu>-m-2)$

172. $\displaystyle\int_0^\infty x^{\nu}e^{-ax}J_\nu(bx)dx = \frac{(2b)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{\nu+\frac{1}{2}}}$ $\left(\mathrm{Re}\,\nu>-\frac{1}{2},\ \mathrm{Re}\,a>|\,\mathrm{Im}\,b\,|\right)$

[这里，$\Gamma(z)$ 为伽马函数(见附录)，以下同]

173. $\int_0^\infty x^{\nu+1}e^{-ax}J_\nu(bx)dx = \dfrac{2a(2b)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{\nu+\frac{3}{2}}} \quad \left(\mathrm{Re}\,\nu > -1,\ \mathrm{Re}\,a > |\,\mathrm{Im}\,b\,|\right)$

Ⅱ.2.4.7 贝塞尔函数与对数函数或双曲函数组合的积分

174. $\int_0^\infty \ln x J_0(ax)dx = -\dfrac{1}{a}[\ln(2a)+\gamma]$

[这里，γ 为欧拉常数(见附录)，以下同]

175. $\int_0^\infty \ln x J_1(ax)dx = -\dfrac{1}{a}\left(\ln\dfrac{a}{2}+\gamma\right)$

176. $\int_0^\infty K_\nu(bx)\cosh ax dx = \dfrac{\pi}{2}\dfrac{\cos\left(\nu\arcsin\frac{a}{b}\right)}{\sqrt{b^2-a^2}\cos\frac{\nu\pi}{2}} \quad (\mathrm{Re}\,b > |\,\mathrm{Re}\,a\,|,\ |\,\mathrm{Re}\,\nu\,| < 1)$

Ⅱ.2.5 勒让德(Legendre)函数和连带勒让德函数的定积分

Ⅱ.2.5.1 连带勒让德函数的积分

177. $\int_{-1}^1 P_n^m(x)P_k^m(x)dx = \begin{cases} 0 & (n \neq k) \\ \dfrac{2(n+m)!}{(2n+1)(n-m)!} & (n = k) \end{cases}$

[这里，$P_n^m(x)$ 为连带勒让德函数(见附录)，以下同]

178. $\int_0^1 \dfrac{[P_n^m(x)]^2}{1-x^2}dx = \dfrac{(n+m)!}{2m(n-m)!} \quad (0 < m \leqslant n)$

179. $\int_{-1}^1 \dfrac{P_n^m(x)P_n^k(x)}{1-x^2}dx = 0 \quad (0 \leqslant m \leqslant n,\ 0 \leqslant k \leqslant n;\ m \neq k)$

Ⅱ.2.5.2 勒让德多项式与代数函数组合的积分

180. $\int_{-1}^1 P_n(x)P_m(x)dx = \begin{cases} 0 & (m \neq n) \\ \dfrac{2}{2n+1} & (m = n) \end{cases}$

[这里，$P_n(x)$ 为勒让德多项式(见附录)，以下同]

181. $\int_0^{2\pi} P_{2n}(\cos\varphi)\mathrm{d}\varphi = 2\pi\left[\binom{2n}{n}2^{-2n}\right]^2$

182. $\int_{-1}^{1} x^m P_n(x)\mathrm{d}x = \begin{cases} 0 & (m < n) \\ \dfrac{m![1+(-1)^{m-n}]}{(m-n)!!(m+n+1)!!} & (m \geqslant n) \end{cases}$

183. $\int_0^1 x^\lambda P_{2m}(x)\mathrm{d}x = \dfrac{(-1)^m\Gamma\left(m-\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}\right)}{2\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(m+\frac{3}{2}+\frac{\lambda}{2}\right)} \quad (\mathrm{Re}\,\lambda > -1)$

[这里，$\Gamma(z)$为伽马函数(见附录)，以下同]

184. $\int_0^1 x^\lambda P_{2m+1}(x)\mathrm{d}x = \dfrac{(-1)^m\Gamma\left(m+\frac{1}{2}-\frac{\lambda}{2}\right)\Gamma\left(1+\frac{\lambda}{2}\right)}{2\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}\right)\Gamma\left(m+2+\frac{\lambda}{2}\right)} \quad (\mathrm{Re}\,\lambda > -2)$

185. $\int_{-1}^{1}(1-x^2)^n P_{2m}(x)\mathrm{d}x = \dfrac{2n^2}{(n-m)(2m+2n+1)}\int_{-1}^{1}(1-x^2)^{n-1}P_{2m}(x)\mathrm{d}x$

$(m < n)$

186. $\int_{-1}^{1}(1+x)^{m+n}P_m(x)P_n(x)\mathrm{d}x = \dfrac{2^{m+n+1}[(m+n)!]^4}{(m!n!)^2(2m+2n+1)!}$

187. $\int_{-1}^{1}(1+x)^{m-n-1}P_m(x)P_n(x)\mathrm{d}x = 0 \quad (m > n)$

188. $\int_0^1 x^2 P_{n+1}(x)P_{n-1}(x)\mathrm{d}x = \dfrac{n(n+1)}{(2n-1)(2n+1)(2n+3)}$

189. $\int_{-1}^{1}\dfrac{1}{z-x}[P_n(x)P_{n-1}(z) - P_{n-1}(x)P_n(z)]\mathrm{d}x = -\dfrac{2}{n}$

190. $\int_{-1}^{1}(1-x)^{-\frac{1}{2}}P_n(x)\mathrm{d}x = \dfrac{2\sqrt{2}}{2n+1}$

191. $\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}P_{2m}(x)\mathrm{d}x = \left[\dfrac{\Gamma\left(\frac{1}{2}+m\right)}{m!}\right]^2$

192. $\int_{-1}^{1}x(1-x^2)^{-\frac{1}{2}}P_{2m+1}(x)\mathrm{d}x = \dfrac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!}$

193. $\int_{-1}^{1}(1+px^2)^{-m-\frac{3}{2}}P_{2m}(x)\mathrm{d}x = \dfrac{2}{2m+1}(-p)^m(1+p)^{-m-\frac{1}{2}} \quad (|p|<1)$

Ⅱ.2.5.3 勒让德多项式与其他初等函数组合的积分

194. $\int_0^\infty P_n(\mathrm{e}^{-x})\mathrm{e}^{-ax}\mathrm{d}x = \dfrac{(a-1)(a-2)\cdots(a-n+1)}{(a+n)(a+n-2)\cdots(a-n+2)} \quad (\mathrm{Re}\,a > 0,\ n \geqslant 2)$

195. $\int_0^\infty P_{2n}(\cos x)\mathrm{e}^{-ax}\mathrm{d}x = \dfrac{(a^2+1^2)(a^2+3^2)\cdots[a^2+(2n-1)^2]}{a(a^2+2^2)(a^2+4^2)\cdots[a^2+(2n)^2]} \quad (\mathrm{Re}\,a > 0)$

196. $\int_0^{\infty} P_{2n+1}(\cos x)e^{-ax}dx = \frac{a(a^2+2^2)(a^2+4^2)\cdots[a^2+(2n)^2]}{(a^2+1^2)(a^2+3^2)\cdots[a^2+(2n+1)^2]} \quad (\mathrm{Re}\, a>0)$

197. $\int_0^1 P_n(1-2x^2)\sin ax dx = \frac{\pi}{2}\left[J_{n+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 \quad (a>0)$

[这里,$J_\nu(z)$为贝塞尔函数(见附录),以下同]

198. $\int_0^1 P_n(1-2x^2)\cos ax dx = \frac{\pi}{2}(-1)^n J_{n+\frac{1}{2}}\left(\frac{a}{2}\right)J_{-n-\frac{1}{2}}\left(\frac{a}{2}\right) \quad (a>0)$

199. $\int_0^{\pi} P_m(\cos\theta)\sin n\theta d\theta$

$$= \begin{cases} \frac{2(n-m+1)(n-m+3)\cdots(n+m-1)}{(n-m)(n-m+2)\cdots(n+m)} & (n>m,\ n+m\ \text{为奇数}) \\ 0 & (n\leqslant m,\ n+m\ \text{为偶数}) \end{cases}$$

200. $\int_0^{2\pi} P_{2m+1}(\cos\theta)\cos\theta d\theta = \frac{\pi}{2^{4m+1}}\binom{2m}{m}\binom{2m+2}{m+1}$

201. $\int_0^{\pi} P_n(1-2\sin^2 x\sin^2\theta)\sin x dx = \frac{2\sin(2n+1)\theta}{(2n+1)\sin\theta}$

202. $\int_0^1 P_{2n+1}(x)\sin ax\frac{dx}{\sqrt{x}} = (-1)^{n+1}\sqrt{\frac{\pi}{2a}}J_{2n+\frac{3}{2}}(a) \quad (a>0)$

203. $\int_{-1}^1 P_n(x)(\cosh 2p - x)^{-\frac{1}{2}}dx = \frac{2\sqrt{2}}{2n+1}\exp[-(2n+1)p] \quad (p\geqslant 0)$

Ⅱ.2.6 正交多项式的定积分

Ⅱ.2.6.1 埃尔米特(Hermite)多项式的积分

204. $\int_0^x H_n(y)dy = \frac{1}{2(n+1)}[H_{n+1}(x) - H_{n+1}(0)]$

[这里,$H_n(x)$为埃尔米特多项式(见附录),以下同]

205. $\int_0^x e^{-y^2}H_n(y)dy = H_{n-1}(0) - e^{-x^2}H_{n-1}(x)$

206. $\int_{-\infty}^{\infty} e^{-(x-y)^2}H_n(x)dx = \sqrt{\pi}2^n y^n$

207. $\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = \begin{cases} 0 & (m\neq n) \\ 2^n n!\sqrt{\pi} & (m=n) \end{cases}$

208. $\int_{-\infty}^{\infty} e^{-x^2}H_m(ax)H_n(x)dx = 0 \quad (m<n)$

209. $\int_{-\infty}^{\infty} e^{-x^2}H_{2m+n}(ax)H_n(x)dx = \frac{(2m+n)!}{m!}\sqrt{\pi}(2a)^n(a^2-1)^m$

210. $\int_{-\infty}^{\infty} e^{-2x^2} H_m(x) H_n(x) dx = (-1)^{\frac{1}{2}(m+n)} 2^{\frac{1}{2}(m+n-1)} \Gamma\left(\frac{m+n+1}{2}\right)$

($m+n$ 为偶数)

[这里,$\Gamma(z)$为伽马函数(见附录),以下同]

211. $\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x) dx = 2^n \sqrt{\pi} m! y^{n-m} L_m^{n-m}(-2y^2) \quad (m \leqslant n)$

[这里,$L_n^{n-m}(x)$为连带拉盖尔多项式(见附录),以下同]

212. $\int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z) dx = 2^n \sqrt{\pi} m! z^{n-m} L_m^{n-m}(-2yz) \quad (m \leqslant n)$

213. $\int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x) dx = i^n \sqrt{2\pi} e^{-\frac{y^2}{2}} H_n(y)$

214. $\int_0^{\infty} x^{a-1} e^{-bx} H_n(x) dx = 2^n \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n!\Gamma(a+n-2m)}{m!(n-2m)!} (-1)^m 2^{-2m} b^{2m-a-n}$

(如果 n 是偶数,则 $\mathrm{Re}\, a > 0$;如果 n 是奇数,则 $\mathrm{Re}\, a > -1$; $\mathrm{Re}\, b > 0$)

215. $\int_{-\infty}^{\infty} e^{-x^2} H_{2m}(yx) dx = \frac{(2m)!}{m!} \sqrt{\pi} (y^2-1)^m$

216. $\int_{-\infty}^{\infty} x e^{-x^2} H_{2m+1}(yx) dx = \sqrt{\pi} \frac{(2m+1)!}{m!} y (y^2-1)^m$

217. $\int_{-\infty}^{\infty} x^n e^{-x^2} H_n(yx) dx = \sqrt{\pi} n! P_n(y)$

[这里,$P_n(y)$为勒让德多项式(见附录)]

218. $\int_0^{\infty} x^{-\frac{1}{2}} e^{-px} H_{2n}\left(\sqrt{x}\right) dx = (-1)^n 2^n (2n-1)!! \sqrt{\pi} (p-1)^n p^{-n-\frac{1}{2}}$

Ⅱ.2.6.2 拉盖尔(Laguerre)多项式的积分

219. $\int_0^t L_n(x) dx = L_n(t) - \frac{1}{n+1} L_{n+1}(t)$

[这里,$L_n(x)$为拉盖尔多项式(见附录),以下同]

220. $\int_0^t L_m(x) L_n(t-x) dx = L_{m+n}(t) - L_{m+n+1}(t)$

221. $\int_0^{\infty} e^{-bx} L_n(x) dx = (b-1)^n b^{-n-1} \quad (\mathrm{Re}\, b > 0)$

222. $\int_0^{\infty} e^{-bx} L_n(\lambda x) L_n(\mu x) dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}} P_n\left[\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right]$

($\mathrm{Re}\, b > 0$)

[这里,$P_n(x)$为勒让德多项式(见附录)]

Ⅱ.2.7 δ函数的积分

223. $\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x = 1$

[这里,$\delta(x)$为δ函数(见附录),以下同]

224. $\int_{-\infty}^{\infty}\delta(a-x)\delta(x-b)\mathrm{d}x = \delta(a-b)$

225. $\int_{-\infty}^{\infty}f(x)\delta(x-a)\mathrm{d}x = f(a)$

226. $\int_{-\infty}^{\infty}f(x)\frac{\mathrm{d}^m\delta(x)}{\mathrm{d}x^m}\mathrm{d}x = (-1)^m\frac{\mathrm{d}^m f(0)}{\mathrm{d}x^m}$

227. $\int_{-\infty}^{\infty}f(x)\delta[\varphi(x)]\mathrm{d}x = \sum_i\frac{f(x_i)}{|\varphi'(x_i)|}$

{这里,要求方程 $\varphi(x)=0$ 只有单根[即 $\varphi(x)$ 的零点],公式的右边表示对 $\varphi(x)$的所有零点 $x_i(i=1,2,3,\cdots)$求和}

Ⅲ 积分变换表

Ⅲ.1 拉普拉斯(Laplace)变换

拉普拉斯变换定义为

$$F(p) = L[f(x)] = \int_0^{\infty} f(x)\mathrm{e}^{-px}\mathrm{d}x \quad (\mathrm{Re}\ p > c,\ c\ 为常数,\ c \geqslant 0)$$

函数 $f(x)$和 $F(p)$称为拉普拉斯变换对. 它的逆变换为

$$f(x) = L^{-1}[F(p)] = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} F(p)\mathrm{e}^{px}\mathrm{d}p \quad (\sigma > c).$$

拉普拉斯变换表

编号	$f(x)$	$F(p)$
1	1	$\frac{1}{p}$
2	x	$\frac{1}{p^2}$
3	$x^n \quad (n=0,1,2,\cdots)$	$\frac{n!}{p^{n+1}} \quad (\mathrm{Re}\ p>0)$
4	$x^{\nu} \quad (\nu>-1)$	$\frac{\Gamma(\nu+1)}{p^{\nu+1}} \quad (\mathrm{Re}\ p>0)$
5	$x^{n-\frac{1}{2}}$	$\frac{1}{p^{n+\frac{1}{2}}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\cdots\cdot\frac{n-1}{2} \quad (\mathrm{Re}\ p>0)$
6	$\sqrt{x}$	$\frac{\sqrt{\pi}}{2}\cdot\frac{1}{p^{\frac{3}{2}}}$
7	$\frac{1}{\sqrt{x}}$	$\sqrt{\frac{\pi}{p}}$
8	$\frac{\sqrt{x}}{x+a} \quad (\lvert\arg a\rvert<\pi)$	$\sqrt{\frac{\pi}{p}}-\pi\mathrm{e}^{ap}\sqrt{a} \quad (\mathrm{Re}\ p>0)$

续表

编号	$f(x)$	$F(p)$
9	$\begin{cases} x & (0<x<1) \\ 1 & (x>1) \end{cases}$	$\frac{1-e^{-p}}{p^2}$ (Re $p>0$)
10	e^{-ax}	$\frac{1}{p+a}$ (Re $p>-$Re a)
11	xe^{-ax}	$\frac{1}{(p+a)^2}$ (Re $p>-$Re a)
12	$x^{\nu-1}e^{-ax}$ (Re $\nu>0$)	$\frac{\Gamma(\nu)}{(p+a)^\nu}$ (Re $p>-$Re a)
13	$xe^{-\frac{x^2}{4a}}$ (Re $a>0$)	$2a-2\pi^{\frac{1}{2}}a^{\frac{3}{2}}pe^{ap^2}\operatorname{erfc}(pa^{\frac{1}{2}})$
14	$\exp(-ae^x)$ (Re $a>0$)	$a^p\Gamma(-p,a)$
15	$\ln x$	$-\frac{1}{p}(\gamma+\ln p)$ (Re $p>0$，γ 为欧拉常数)
16	$\ln(1+ax)$ ($\lvert\arg a\rvert<\pi$)	$-\frac{1}{p}e^{\frac{p}{a}}\operatorname{Ei}\left(-\frac{p}{a}\right)$ (Re $p>0$)
17	$\frac{\ln x}{\sqrt{x}}$	$-\sqrt{\frac{\pi}{p}}\ln(4\gamma p)$ (Re $p>0$)
18	$\sin ax$	$\frac{a}{p^2+a^2}$ (Re $p>\lvert$Im $a\rvert$)
19	$\cos ax$	$\frac{p}{p^2+a^2}$ (Re $p>\lvert$Im $a\rvert$)
20	$\sinh ax$	$\frac{a}{p^2-a^2}$ (Re $p>\lvert$Re $a\rvert$)
21	$\cosh ax$	$\frac{p}{p^2-a^2}$ (Re $p>\lvert$Re $a\rvert$)
22	$x\sin ax$	$\frac{2ap}{(p^2+a^2)^2}$
23	$x\cos ax$	$\frac{p^2-a^2}{(p^2+a^2)^2}$
24	$x\sinh ax$	$\frac{2ap}{(p^2-a^2)^2}$
25	$x\cosh ax$	$\frac{p^2+a^2}{(p^2-a^2)^2}$
26	$x^{\nu-1}\sin ax$ (Re $\nu>-1$)	$\frac{i\Gamma(\nu)}{2}\left[\frac{1}{(p+ia)^\nu}-\frac{1}{(p-ia)^\nu}\right]$
27	$x^{\nu-1}\cos ax$ (Re $\nu>-1$)	$\frac{\Gamma(\nu)}{2}\left[\frac{1}{(p+ia)^\nu}+\frac{1}{(p-ia)^\nu}\right]$
28	$x^{\nu-1}\sinh ax$ (Re $\nu>-1$)	$\frac{\Gamma(\nu)}{2}\left[\frac{1}{(p-a)^\nu}-\frac{1}{(p+a)^\nu}\right]$ (Re $p>\lvert$Re $a\rvert$)
29	$x^{\nu-1}\cosh ax$ (Re $\nu>0$)	$\frac{\Gamma(\nu)}{2}\left[\frac{1}{(p-a)^\nu}+\frac{1}{(p+a)^\nu}\right]$ (Re $p>\lvert$Re $a\rvert$)

续表

编号	$f(x)$	$F(p)$
30	$e^{-bx}\sin ax$	$\frac{a}{(p+b)^2+a^2}$
31	$e^{-bx}\cos ax$	$\frac{p+b}{(p+b)^2+a^2}$
32	$e^{-bx}\sin(ax+c)$	$\frac{(p+b)\sin c+a\cos c}{(p+b)^2+a^2}$
33	$e^{-bx}\cos(ax+c)$	$\frac{(p+b)\cos c-a\sin c}{(p+b)^2+a^2}$
34	$\sin^2 ax$	$\frac{2a^2}{p(p^2+4a^2)}$
35	$\cos^2 ax$	$\frac{p^2+2a}{p(p^2+4a^2)}$
36	$\sin ax\sin bx$	$\frac{2abp}{[p^2+(a+b)^2][p^2+(a-b)^2]}$
37	$e^{ax}-e^{bx}$	$\frac{a-b}{(p-a)(p-b)}$
38	$ae^{ax}-be^{bx}$	$\frac{(a-b)p}{(p-a)(p-b)}$
39	$\frac{1}{a}\sin ax-\frac{1}{b}\sin bx$	$\frac{b^2-a^2}{(p^2+a^2)(p^2+b^2)}$
40	$\cos ax-\cos bx$	$\frac{(b^2-a^2)p}{(p^2+a^2)(p^2+b^2)}$
41	$\frac{1}{a^3}(ax-\sin ax)$	$\frac{1}{p^2(p^2+a^2)}$
42	$\frac{1}{a^4}(\cos ax-1)+\frac{1}{2a^2}x^2$	$\frac{1}{p^3(p^2+a^2)}$
43	$\frac{1}{a^4}(\cosh ax-1)-\frac{1}{2a^2}x^2$	$\frac{1}{p^2(p^2-a^2)}$
44	$\frac{1}{2a^3}(\sin ax-ax\cos ax)$	$\frac{1}{(p^2+a^2)^2}$
45	$\frac{1}{2a}(\sin ax+ax\cos ax)$	$\frac{p^2}{(p^2+a^2)^2}$
46	$\frac{1}{a^4}(1-\cos ax)-\frac{x}{2a^3}\sin ax$	$\frac{1}{p(p^2+a^2)^2}$
47	$(1-ax)e^{-ax}$	$\frac{p}{(p+a)^2}$
48	$x\left(1-\frac{a}{2}x\right)e^{-ax}$	$\frac{p}{(p+a)^3}$
49	$\frac{1}{a}(1-e^{-ax})$	$\frac{1}{p(p+a)}$
50	$\frac{1}{ab}+\frac{1}{b-a}\left(\frac{e^{-bx}}{b}-\frac{e^{-ax}}{a}\right)$	$\frac{1}{p(p+a)(p+b)}$

续表

编号	$f(x)$	$F(p)$
51	$\sin ax\cosh ax-\cos ax\sinh ax$	$\frac{4a^3}{p^4+4a^4}$
52	$\frac{1}{2a^2}\sin ax\sinh ax$	$\frac{p}{p^4+4a^4}$
53	$\frac{1}{2a^3}(\sinh ax-\sin ax)$	$\frac{1}{p^4-a^4}$
54	$\frac{1}{2a^2}(\cosh ax-\cos ax)$	$\frac{p}{p^4-a^4}$
55	$\frac{1}{\sqrt{\pi x}}$	$\frac{1}{\sqrt{p}}$
56	$2\sqrt{\frac{x}{\pi}}$	$\frac{1}{p\sqrt{p}}$
57	$\frac{1}{\sqrt{\pi x}}e^{ax}(1+2ax)$	$\frac{p}{(p-a)\sqrt{(p-a)}}$
58	$\frac{1}{2\sqrt{\pi x^3}}(e^{bx}-e^{ax})$	$\sqrt{p-a}-\sqrt{p-b}$
59	$\frac{1}{\sqrt{\pi x}}\cos(2\sqrt{ax})$	$\frac{1}{\sqrt{p}}e^{-\frac{a}{p}}$
60	$\frac{1}{\sqrt{\pi x}}\cosh(2\sqrt{ax})$	$\frac{1}{\sqrt{p}}e^{\frac{a}{p}}$
61	$\frac{1}{\sqrt{\pi x}}\sin(2\sqrt{ax})$	$\frac{1}{p\sqrt{p}}e^{-\frac{a}{p}}$
62	$\frac{1}{\sqrt{\pi x}}\sinh(2\sqrt{ax})$	$\frac{1}{p\sqrt{p}}e^{\frac{a}{p}}$
63	$\frac{1}{x}(e^{bx}-e^{ax})$	$\ln\frac{p-a}{p-b}$
64	$\frac{2}{x}\sinh ax$	$\ln\frac{p+a}{p-a}$
65	$\frac{2}{x}(1-\cos ax)$	$\ln\frac{p^2+a^2}{p^2}$
66	$\frac{2}{x}(1-\cosh ax)$	$\ln\frac{p^2-a^2}{p^2}$
67	$\frac{1}{x}\sin ax$	$\arctan\frac{a}{p}$
68	$\frac{1}{x}(\cosh ax-\cos ax)$	$\ln\sqrt{\frac{p^2+b^2}{p^2-a^2}}$
69	$\mathrm{Si}(x)\equiv\int_0^x\frac{\sin\xi}{\xi}d\xi$	$\frac{1}{p}\operatorname{arccot}p\quad(\mathrm{Re}\ p>0)$
70	$\mathrm{Ci}(x)\equiv-\int_x^\infty\frac{\cos\xi}{\xi}d\xi$	$-\frac{1}{2p}\ln(1+p^2)\quad(\mathrm{Re}\ p>0)$
71	$\frac{1}{\pi x}\sin(2a\sqrt{x})$	$\mathrm{erf}\left(\frac{a}{\sqrt{p}}\right)$

续表

编号	$f(x)$	$F(p)$
72	$\frac{1}{\sqrt{\pi x}}e^{-2a\sqrt{x}}\quad(a>0)$	$\frac{1}{\sqrt{p}}e^{\frac{a^2}{p}}\operatorname{erfc}\left(\frac{a}{\sqrt{p}}\right)$
73	$\Phi(a\sqrt{x})$	$\frac{a}{p\sqrt{p+a^2}}\quad(\operatorname{Re}p>0)$
74	$\operatorname{erfc}(a\sqrt{x})\equiv 1-\Phi(a\sqrt{x})$	$1-\frac{a}{\sqrt{p+a^2}}\quad(\operatorname{Re}p>0)$
75	$\operatorname{erfc}\left(\frac{a}{\sqrt{x}}\right)$	$\frac{1}{p}e^{-2a\sqrt{p}}\quad(\operatorname{Re}p>0)$
76	$\frac{1}{\sqrt{x}}e^{-\frac{a^2}{4x}}\quad(a>0)$	$\sqrt{\frac{\pi}{p}}e^{-a\sqrt{p}}$
77	$\operatorname{erf}\left(\frac{x}{2a}\right)\quad(a>0)$	$\frac{1}{p}e^{a^2p^2}\operatorname{erfc}(ap)$
78	$\frac{1}{\sqrt{\pi(x+a)}}\quad(a>0)$	$\frac{1}{\sqrt{p}}e^{ap}\operatorname{erfc}(\sqrt{ap})$
79	$\frac{1}{\sqrt{a}}\operatorname{erf}(\sqrt{ax})$	$\frac{1}{p\sqrt{p+a}}$
80	$\frac{1}{\sqrt{a}}e^{ax}\operatorname{erf}(\sqrt{ax})$	$\frac{1}{\sqrt{p}(p-a)}$
81	$\lvert\cos ax\rvert\quad(a>0)$	$\frac{1}{p^2+a^2}\left(p+\operatorname{arcosh}\frac{p\pi}{2a}\right)$
82	$\lvert\sin ax\rvert\quad(a>0)$	$\frac{a}{p^2+a^2}\coth\frac{p\pi}{2a}$
83	$J_0(ax)$	$\frac{1}{\sqrt{p^2+a^2}}$
84	$I_0(ax)$	$\frac{1}{\sqrt{p^2-a^2}}$
85	$J_\nu(ax)\quad(\operatorname{Re}\nu>-1)$	$\frac{1}{\sqrt{p^2+a^2}}\frac{a^\nu}{\left(p+\sqrt{p^2+a^2}\right)^\nu}\quad(\operatorname{Re}p>\lvert\operatorname{Im}a\rvert)$
86	$xJ_\nu(ax)\quad(\operatorname{Re}\nu>-2)$	$\frac{p+\nu\sqrt{p^2+a^2}}{(p^2+a^2)^{\frac{3}{2}}}\frac{a^\nu}{\left(p+\sqrt{p^2+a^2}\right)^\nu}$ $(\operatorname{Re}p>\lvert\operatorname{Im}a\rvert)$
87	$\frac{J_\nu(ax)}{x}\quad(\operatorname{Re}\nu>0)$	$\frac{1}{\nu}\frac{a^\nu}{\left(p+\sqrt{p^2+a^2}\right)^\nu}\quad(\operatorname{Re}p>\lvert\operatorname{Im}a\rvert)$
88	$x^nJ_n(ax)$	$\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)a^n}{(p^2+a^2)^{n+\frac{1}{2}}}\quad(\operatorname{Re}p>\lvert\operatorname{Im}a\rvert)$
89	$x^\nu J_\nu(ax)\quad\left(\operatorname{Re}\nu>-\frac{1}{2}\right)$	$\frac{(2a)^\nu}{\sqrt{\pi}(p^2+a^2)^{\nu+\frac{1}{2}}}\Gamma\left(\nu+\frac{1}{2}\right)\quad(\operatorname{Re}p>\lvert\operatorname{Im}a\rvert)$

续表

编号	$f(x)$	$F(p)$
90	$I_\nu(ax)$ $(\mathrm{Re}\ \nu>-1)$	$\frac{1}{\sqrt{p^2-a^2}}\frac{a^\nu}{\left(p+\sqrt{p^2-a^2}\right)^\nu}$ $(\mathrm{Re}\ p>\lvert\mathrm{Re}\ a\rvert)$
91	$x^\nu I_\nu(ax)$ $\left(\mathrm{Re}\ \nu>-\frac{1}{2}\right)$	$\frac{(2a)^\nu}{\sqrt{\pi}(p^2-a^2)^{\nu+\frac{1}{2}}}\Gamma\left(\nu+\frac{1}{2}\right)$ $(\mathrm{Re}\ p>\lvert\mathrm{Re}\ a\rvert)$
92	$\frac{I_\nu(ax)}{x}$ $(\mathrm{Re}\ \nu>0)$	$\frac{1}{\nu}\frac{a^\nu}{\left(p+\sqrt{p^2-a^2}\right)^\nu}$ $(\mathrm{Re}\ p>\lvert\mathrm{Re}\ a\rvert)$
93	$\delta(x)$（狄拉克 δ 函数）	1
94	$\delta'(x)$	p
95	$\delta(x-a)$ $(a>0)$	e^{-ap}
96	$\delta'(x-a)$ $(a>0)$	pe^{-ap}
97	$\delta^{(k)}(x-a)$	$p^k e^{-ap}$ $(-\infty<p<\infty)$
98	$\sum_{m=1}^{\infty}\delta(x-ma)$	$-\frac{1}{1-e^{ap}}$ $(\mathrm{Re}\ p>0)$
99	x_+^λ	$\frac{\Gamma(\lambda+1)}{p^{\lambda+1}}$ $(\lambda\neq-1,-2,\cdots;\ \mathrm{Re}\ p>0)$
100	x_+^{-k}	$-\frac{(-p)^{k-1}}{(k-1)!}[\ln p-\psi(k)]$ $(k=2,3,\cdots;\ \mathrm{Re}\ p>0)$

Ⅲ.2 傅里叶(Fourier)变换

一个函数 $f(t)$ 的傅里叶变换定义为

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt$$

式中，$e^{i\omega t}$ 称为变换核. 它的逆变换为

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega t}d\omega$$

傅里叶变换表

编号	$f(t)$	$F(\omega)$
1	$e^{-a\lvert t\rvert}\quad(a>0)$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$
2	$te^{-a\lvert t\rvert}\quad(a>0)$	$\sqrt{\frac{2}{\pi}}\frac{2ia\omega}{(a^2+\omega^2)^2}$
3	$\lvert t\rvert e^{-a\lvert t\rvert}\quad(a>0)$	$\sqrt{\frac{2}{\pi}}\frac{(a^2-\omega^2)}{(a^2+\omega^2)^2}$
4	$\frac{e^{-a\lvert t\rvert}}{\sqrt{\lvert t\rvert}}\quad(a>0)$	$\frac{\sqrt{a+\sqrt{a^2+\omega^2}}}{\sqrt{a^2+\omega^2}}$
5	$\frac{\operatorname{sgn}t\, e^{-a\lvert t\rvert}}{\sqrt{\lvert t\rvert}}\quad(a>0)$	$\frac{i\operatorname{sgn}\omega\ \sqrt{\sqrt{a^2+\omega^2}-a}}{\sqrt{a^2+\omega^2}}$
6	$e^{-a^2t^2}\quad(a>0)$	$\frac{\exp\left(-\frac{\omega^2}{4a^2}\right)}{a\sqrt{2}}$
7	$e^{-b\sqrt{a^2+t^2}}\quad(a>0,\ b>0)$	$\sqrt{\frac{2}{\pi}}\frac{ab}{\sqrt{b^2+\omega^2}}K_1(a\ \sqrt{b^2+\omega^2})$
8	$\frac{e^{-b\sqrt{a^2+t^2}}}{\sqrt{a^2+t^2}}\quad(a>0,\ b>0)$	$\sqrt{\frac{2}{\pi}}K_0(a\ \sqrt{b^2+\omega^2})$
9	$\frac{1}{a^2+t^2}\quad(\operatorname{Re}a>0)$	$\sqrt{\frac{\pi}{2}}\frac{1}{ae^{a\lvert\omega\rvert}}$
10	$\frac{t}{(a^2+t^2)^2}\quad(\operatorname{Re}a>0)$	$i\sqrt{\frac{\pi}{2}}\frac{\omega}{2ae^{a\lvert\omega\rvert}}$
11	$\frac{1}{\sqrt{a^2+t^2}}$	$\sqrt{\frac{2}{\pi}}K_0(a\lvert\omega\rvert)$
12	$\frac{1}{\sqrt{a^2-t^2}}\quad(\lvert t\rvert<a)$	$\sqrt{\frac{\pi}{2}}J_0(a\lvert\omega\rvert)$
13	$\frac{1}{(a^2+t^2)^{\nu+\frac{1}{2}}}\quad\left(\operatorname{Re}\nu>-\frac{1}{2}\right)$	$\frac{\sqrt{2}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left\lvert\frac{\omega}{2a}\right\rvert^{\nu}K_\nu(a\lvert\omega\rvert)$
14	$\begin{cases}\frac{1}{(a^2-t^2)^{\nu+\frac{1}{2}}} & \left(\lvert t\rvert<a,\ \operatorname{Re}\nu<\frac{1}{2}\right)\\ 0 & \left(\lvert t\rvert>a,\ \operatorname{Re}\nu<\frac{1}{2}\right)\end{cases}$	$\frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\sqrt{2}}\left\lvert\frac{\omega}{2a}\right\rvert^{\nu}J_{-\nu}(a\lvert\omega\rvert)$
15	$\sin at^2$	$\frac{1}{\sqrt{2a}}\sin\left(\frac{\omega^2}{4a}+\frac{\pi}{4}\right)$
16	$\cos at^2$	$\frac{1}{\sqrt{2a}}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$

续表

编号	$f(t)$	$F(\omega)$
17	$\frac{\sin at}{t}$	$\begin{cases}\sqrt{\frac{\pi}{2}} & (\lvert\omega\rvert<a)\\ 0 & (\lvert\omega\rvert>a)\end{cases}$
18	$\frac{t}{\sinh t}$	$\sqrt{\frac{2}{\pi^3}}\frac{e^{\omega\pi}}{(1+e^{\omega\pi})^2}$
19	$\frac{\sin at}{\sqrt{\lvert t\rvert}}$	$\frac{i}{2}\left(\frac{1}{\sqrt{\lvert a+\omega\rvert}}-\frac{1}{\sqrt{\lvert a-\omega\rvert}}\right)$
20	$\frac{\cos at}{\sqrt{\lvert t\rvert}}$	$\frac{1}{2}\left(\frac{1}{\sqrt{\lvert a+\omega\rvert}}+\frac{1}{\sqrt{\lvert a-\omega\rvert}}\right)$
21	$\frac{\sin^2 at}{t^2}$ $(a>0)$	$\begin{cases}\sqrt{\frac{\pi}{2}}\left(a-\frac{\lvert\omega\rvert}{2}\right) & (\lvert\omega\rvert<2a)\\ 0 & (\lvert\omega\rvert>2a)\end{cases}$
22	$\frac{\sinh at}{\sinh bt}$ $(0<a<b)$	$\sqrt{\frac{\pi}{2}}\frac{\sin\frac{a\pi}{b}}{b\left(\cosh\frac{\omega\pi}{b}+\cos\frac{a\pi}{b}\right)}$
23	$\frac{\cosh at}{\sinh bt}$ $(0<a<b)$	$i\sqrt{\frac{\pi}{2}}\frac{\sinh\frac{\omega\pi}{b}}{b\left(\cosh\frac{\omega\pi}{b}+\cos\frac{a\pi}{b}\right)}$
24	$t^\nu \operatorname{sgn} t$ （$\nu<-1$，非整数）	$\sqrt{\frac{2}{\pi}}\frac{\nu!}{(-i\omega)^{1+\nu}}$
25	$\lvert t\rvert^\nu$ （$\nu<-1$，非整数）	$-\sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu+1)}{\lvert\omega\rvert^{\nu+1}}\sin\frac{\nu\pi}{2}$
26	$\frac{1}{\lvert t\rvert^a}$ $(0<\operatorname{Re} a<1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(1-a)}{\lvert\omega\rvert^{1-a}}\sin\frac{a\pi}{2}$
27	$\lvert t\rvert^\nu \operatorname{sgn} t$ （$\nu<-1$，非整数）	$i\operatorname{sgn}\omega\cdot\sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu+1)}{\lvert\omega\rvert^{\nu+1}}\cos\frac{\nu\pi}{2}$
28	$e^{-at}\ln\lvert 1-e^{-t}\rvert$ $(-1<\operatorname{Re} a<0)$	$\sqrt{\frac{\pi}{2}}\frac{\cot(a\pi-i\omega\pi)}{a-i\omega}$
29	$e^{-at}\ln(1+e^{-t})$ $(-1<\operatorname{Re} a<0)$	$\sqrt{\frac{\pi}{2}}\frac{\csc(a\pi-i\omega\pi)}{a-i\omega}$
30	$J_0(\sqrt{b}\sqrt{a^2-t^2})H(a^2-t^2)$	$\sqrt{\frac{2}{\pi}}\frac{\sin(a\sqrt{\omega^2+b})}{\sqrt{\omega^2+b}}$ $(a>0,\ b>0)$

续表

编号	$f(t)$	$F(\omega)$
31	$J_0(\sqrt{b}\sqrt{a^2+t^2})$	$\sqrt{\frac{2}{\pi}}\frac{\cos(a\sqrt{b-\omega^2})}{\sqrt{b-\omega^2}}H(b-\omega^2)\quad(a\geqslant 0,\ b>0)$
32	$\frac{\cosh(\sqrt{b}\sqrt{a^2-t^2})}{\sqrt{a^2-t^2}}H(a^2-t^2)$	$\sqrt{\frac{\pi}{2}}J_0(a\sqrt{\omega^2-b})H(\omega^2-b)\quad(a>0,\ b\geqslant 0)$
33	$e^{-at}H(t)$	$\frac{i}{\sqrt{2\pi}(\omega+ia)}\quad(a>0)$
34	$P_n(t)H(1-t^2)$	$\frac{i^n}{\sqrt{\pi}}J_{n+\frac{1}{2}}(\omega)$
35	1	$\sqrt{2\pi}\delta(\omega)$
36	t	$-i\sqrt{2\pi}\delta'(\omega)$
37	t^n	$(-i)^n\sqrt{2\pi}\delta^{(n)}(\omega)$
38	$\frac{1}{t}$	$\sqrt{\frac{\pi}{2}}i\,\mathrm{sgn}\omega$
39	$\delta(t)$	$\frac{1}{\sqrt{2\pi}}$
40	$\delta(t-\tau)$	$\frac{e^{i\tau\omega}}{\sqrt{2\pi}}$
41	$\delta^{(n)}(t)$	$\frac{(-i\omega)^n}{\sqrt{2\pi}}$
42	$\frac{1}{\lvert t\rvert}$	$\frac{1}{\lvert\omega\rvert}$
43	e^{iat} （a 为实数）	$\sqrt{2\pi}\delta(\omega+a)$
44	$\cos bt$	$\sqrt{\frac{\pi}{2}}[\delta(\omega+b)+\delta(\omega-b)]$
45	$\sin bt$	$-i\sqrt{\frac{\pi}{2}}[\delta(\omega+b)-\delta(\omega-b)]$
46	$\cosh bt$	$\sqrt{\frac{\pi}{2}}[\delta(\omega+ib)+\delta(\omega-ib)]$
47	$\sinh bt$	$\sqrt{\frac{\pi}{2}}[\delta(\omega+ib)-\delta(\omega-ib)]$
48	$H(t)^*$	$\sqrt{\frac{\pi}{2}}\delta(\omega)+\frac{i}{\sqrt{2\pi}}\frac{1}{\omega}$
49	$\mathrm{sgn}\,t$	$\frac{2i}{\sqrt{2\pi}}\frac{1}{\omega}$

续表

编号	$f(t)$	$F(\omega)$
50	$\frac{1}{t}$	$\mathrm{i}\sqrt{\frac{\pi}{2}}\operatorname{sgn}\omega$
51	t^{-m}	$\frac{\mathrm{i}^m\omega^{m-1}}{(m-1)!}\sqrt{\frac{\pi}{2}}\operatorname{sgn}\omega \quad (m=1,2,\cdots)$
52	$(t-a)^{-m}$	$\frac{\mathrm{i}^m\omega^{m-1}}{(m-1)!}\sqrt{\frac{\pi}{2}}\mathrm{e}^{\mathrm{i}a\omega}\operatorname{sgn}\omega$
53	$t_{\pm}{}^{\lambda}$	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\pm\frac{\mathrm{i}(\lambda+1)\pi}{2}}\Gamma(\lambda+1)(\omega\pm\mathrm{i}0)^{-\lambda-1}$ $(\lambda\neq-1,-2,\cdots)$
54	$\frac{t_{\pm}{}^{\lambda}}{\Gamma(\lambda+1)}$	$\frac{\mathrm{e}^{\pm\mathrm{i}\left(\frac{\lambda}{2}+\frac{1}{2}\right)\pi}(\omega\pm\mathrm{i}0)^{-\lambda-1}}{\sqrt{2\pi}}$ $(\lambda\neq-1,-2,\cdots)$
55	$t_{+}{}^{m}$	$\frac{\mathrm{i}^{m+1}}{\sqrt{2\pi}}[m!\,\omega^{-m-1}+(-1)^{m+1}\mathrm{i}\pi\delta^{(m)}(\omega)]$ $(m=1,2,\cdots)$
56	$t_{-}{}^{m}$	$\frac{\mathrm{i}^{m+1}}{\sqrt{2\pi}}[(-1)^{m+1}m!\,\omega^{-m-1}-\mathrm{i}\pi\delta^{(m)}(\omega)]$ $(m=1,2,\cdots)$
57	$\lvert t\rvert^{\lambda}$	$-\frac{2\Gamma(\lambda+1)}{\sqrt{2\pi}\lvert\omega\rvert^{\lambda+1}}\sin\frac{\lambda\pi}{2} \quad (\lambda\neq\pm1,\pm2,\cdots)$
58	$\lvert t\rvert^{\lambda}\operatorname{sgn}t$	$\frac{2\mathrm{i}\Gamma(\lambda+1)}{\sqrt{2\pi}\lvert\omega\rvert^{\lambda+1}}\cos\frac{\lambda\pi}{2}\operatorname{sgn}\omega \quad (\lambda\neq\pm1,\pm2,\cdots)$
59	$\lvert t\rvert^{m}$	$\frac{\mathrm{i}^{m+1}}{\sqrt{2\pi}}\{[1+(-1)^{m+1}]m!\,\omega^{-m-1}+[(-1)^{m+1}-1]\mathrm{i}\pi\delta^{(m)}(\omega)\} \quad (m=0,1,2,\cdots)$
60	$\lvert t\rvert^{m}\operatorname{sgn}t$	$\frac{\mathrm{i}^{m+1}}{\sqrt{2\pi}}\{[1-(-1)^{m+1}]m!\,\omega^{-m-1}+[(-1)^{m+1}+1]\mathrm{i}\pi\delta^{(m)}(\omega)\} \quad (m=0,1,2,\cdots)$
61	$\ln x_{\pm}$	$\pm\frac{\mathrm{i}}{\sqrt{2\pi}(\omega\pm\mathrm{i}0)}\left[\Gamma'(1)\pm\frac{\mathrm{i}\pi}{2}-\ln(\omega\pm\mathrm{i}0)\right]$
62	$\ln\lvert x\rvert$	$-\sqrt{\frac{\pi}{2}}\left[\frac{1}{\lvert\omega\rvert}+2\gamma\delta(\omega)\right]$
63	$(x^2+1)^{\lambda}$	$\frac{\sqrt{2}}{\Gamma(-\lambda)}\left(\frac{\lvert\omega\rvert}{2}\right)^{-\lambda-\frac{1}{2}}\mathrm{N}_{-\lambda-\frac{1}{2}}(\lvert\omega\rvert)$

续表

编号	$f(t)$	$F(\omega)$
64	$(x^2-1)_+^{\lambda}$	$-\frac{\Gamma(\lambda+1)}{\sqrt{2}}\left(\frac{\lvert\omega\rvert}{2}\right)^{-\lambda-\frac{1}{2}}\mathrm{N}_{-\lambda-\frac{1}{2}}(\lvert\omega\rvert)$
65	$(x^2-1)_+^{m}$	$(-1)^m\sqrt{2\pi}\left(1+\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\right)\delta(\omega)$ $+\frac{(-1)^{m+1}}{\sqrt{2}}\left(\frac{\omega}{2}\right)^{-m-\frac{1}{2}}\mathrm{J}_{m+\frac{1}{2}}(\omega)$ $(m=1,2,\cdots)$

* H(t)为赫维赛德(Heaviside)函数,通常也称阶跃函数,它的表达式为

$$\mathrm{H}(t)=\begin{cases}0 & (t\leqslant 0)\\ 1 & (t>0)\end{cases}$$

Ⅲ.3 傅里叶(Fourier)正弦变换

傅里叶正弦变换定义为

$$F_s(\xi)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\xi x\mathrm{d}x$$

它的逆变换为

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_s(\xi)\sin\xi x\mathrm{d}\xi$$

傅里叶正弦变换表

编号	$f(x)$	$F_s(\xi)$
1	$\frac{1}{x}$	$\sqrt{\frac{\pi}{2}}\quad(\xi>0)$
2	$x^{-\nu}\quad(0<\mathrm{Re}\,\nu<2)$	$\sqrt{\frac{2}{\pi}}\xi^{\nu-1}\Gamma(1-\nu)\cos\frac{\nu\pi}{2}\quad(\xi>0)$
3	$x^{-\frac{1}{2}}$	$\xi^{-\frac{1}{2}}\quad(\xi>0)$
4	$x^{-\frac{3}{2}}$	$2\sqrt{\xi}\quad(\xi>0)$
5	$\frac{\sin ax}{x}\quad(a>0)$	$\frac{1}{\sqrt{2\pi}}\ln\left\lvert\frac{\xi+a}{\xi-a}\right\rvert\quad(\xi>0)$

续表

编号	$f(x)$	$F_s(\xi)$
6	$\frac{\sin ax}{x^2}$ $(a>0)$	$\begin{cases} \xi\sqrt{\frac{\pi}{2}} & (0<\xi<a) \\ a\sqrt{\frac{\pi}{2}} & (a<\xi<\infty) \end{cases}$
7	$\sin\frac{a^2}{x}$ $(a>0)$	$a\sqrt{\frac{\pi}{2}}\frac{J_1(2a\sqrt{\xi})}{\sqrt{\xi}}$ $(\xi>0)$
8	$\frac{1}{x}\sin\frac{a^2}{x}$ $(a>0)$	$\sqrt{\frac{\pi}{2}}N_0(2a\sqrt{\xi})+\sqrt{\frac{2}{\pi}}K_0(2a\sqrt{\xi})$ $(\xi>0)$
9	$\frac{1}{x^2}\sin\frac{a^2}{x}$ $(a>0)$	$\sqrt{\frac{\pi}{2}}\frac{\sqrt{\xi}}{a}J_1(2a\sqrt{\xi})$ $(\xi>0)$
10	$\frac{x}{a^2+x^2}$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{\pi}{2}}e^{-a\xi}$ $(\xi>0)$
11	$\frac{x}{(a^2+x^2)^2}$	$\sqrt{\frac{\pi}{8}}\frac{\xi}{a}e^{-a\xi}$ $(\xi>0)$
12	$\frac{1}{x(a^2+x^2)}$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{\pi}{2}}\frac{1-e^{-a\xi}}{a^2}$ $(\xi>0)$
13	e^{-ax} $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}$ $(\xi>0)$
14	xe^{-ax} $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{2}{\pi}}\frac{2a\xi}{(a^2+\xi^2)^2}$ $(\xi>0)$
15	$x^{-1}e^{-ax}$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{2}{\pi}}\arctan\frac{\xi}{a}$ $(\xi>0)$
16	$x^{\nu-1}e^{-ax}$ $(\mathrm{Re}\ a>0,\ \mathrm{Re}\ \nu>-1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu)\sin\left(\nu\arctan\frac{\xi}{a}\right)}{(a^2+\xi^2)^{\frac{\nu}{2}}}$ $(\xi>0)$
17	$\mathrm{csch}\,ax$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{\pi}{2}}\frac{\tanh\frac{\xi\pi}{2a}}{a}$ $(\xi>0)$
18	$\coth\frac{ax}{2}-1$ $(\mathrm{Re}\ a>0)$	$\sqrt{2\pi}\frac{\coth\frac{\xi\pi}{a}}{a}-\xi$ $(\xi>0)$
19	$J_0(ax)$ $(a>0)$	$\begin{cases} 0 & (0<\xi<a) \\ \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\xi^2-a^2}} & (a<\xi<\infty) \end{cases}$

续表

编号	$f(x)$	$F_s(\xi)$
20	$J_\nu(ax)$ $(a>0,\ \mathrm{Re}\ \nu>-2)$	$\begin{cases}\sqrt{\frac{2}{\pi}}\frac{\sin\left(\nu\arcsin\frac{\xi}{a}\right)}{\sqrt{a^2-\xi^2}} & (0<\xi<a)\\ \frac{a^\nu\cos\frac{\nu\pi}{2}}{\sqrt{\xi^2-a^2}(\xi+\sqrt{\xi^2-a^2})^\nu} & (a<\xi<\infty)\end{cases}$
21	$\frac{J_0(ax)}{x}$ $(a>0)$	$\begin{cases}\sqrt{\frac{2}{\pi}}\arcsin\frac{\xi}{a} & (0<\xi<a)\\ \sqrt{\frac{\pi}{2}} & (a<\xi<\infty)\end{cases}$
22	$\frac{J_0(ax)}{x^2+b^2}$ $(a>0,\ \mathrm{Re}\ b>0)$	$\sqrt{\frac{2}{\pi}}\frac{\sinh b\xi\ K_0(ab)}{b}$ $(0<\xi<a)$
23	$\frac{xJ_0(ax)}{x^2+b^2}$ $(a>0,\ \mathrm{Re}\ b>0)$	$\sqrt{\frac{\pi}{2}}\frac{I_0(ab)}{e^{b\xi}}$ $(a<\xi<\infty)$

Ⅲ.4 傅里叶(Fourier)余弦变换

傅里叶余弦变换定义为

$$F_c(\xi)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\xi x\,dx$$

它的逆变换为

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\xi)\cos\xi x\,d\xi$$

傅里叶余弦变换表

编号	$f(x)$	$F_c(\xi)$
1	$x^{-\nu}$ $(0<\mathrm{Re}\ \nu<1)$	$\sqrt{\frac{\pi}{2}}\frac{\xi^{\nu-1}\sec\frac{\nu\pi}{2}}{\Gamma(\nu)}$ $(\xi>0)$
2	$\frac{1}{x^2+a^2}$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{\pi}{2}}\frac{1}{ae^{a\xi}}$ $(\xi>0)$

续表

编号	$f(x)$	$F_c(\xi)$
3	$\frac{1}{(x^2+a^2)^2}$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{\pi}{2}}\frac{1+a\xi}{2a^3\mathrm{e}^{a\xi}}$ $(\xi>0)$
4	$\frac{1}{(x^2+a^2)^{\nu+\frac{1}{2}}}$ $\left(\mathrm{Re}\ a>0,\ \mathrm{Re}\ \nu>-\frac{1}{2}\right)$	$\sqrt{2}\left(\frac{\xi}{2a}\right)^{\nu}\frac{\mathrm{K}_\nu(a\xi)}{\Gamma(\nu+\frac{1}{2})}$ $(\xi>0)$
5	e^{-ax} $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}$ $(\xi>0)$
6	$x\mathrm{e}^{-ax}$ $(\mathrm{Re}\ a>0)$	$\sqrt{\frac{2}{\pi}}\frac{a^2-\xi^2}{(a^2+\xi^2)^2}$ $(\xi>0)$
7	$x^{\nu-1}\mathrm{e}^{-ax}$ $(\mathrm{Re}\ a>0,\ \mathrm{Re}\ \nu>0)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu)\cos\left(\nu\arctan\frac{\xi}{a}\right)}{(a^2+\xi^2)^{\frac{\nu}{2}}}$ $(\xi>0)$
8	$\mathrm{e}^{-a^2x^2}$ $(\mathrm{Re}\ a>0)$	$\frac{1}{\sqrt{2}\lvert a\rvert\exp\left(\frac{\xi^2}{4a^2}\right)}$ $(\xi>0)$
9	$\frac{\sin x}{x\mathrm{e}^x}$	$\frac{\arctan\frac{2}{\xi^2}}{\sqrt{2\pi}}$ $(\xi>0)$
10	$\sin ax^2$ $(a>0)$	$\frac{1}{2\sqrt{a}}\left(\cos\frac{\xi^2}{4a}-\sin\frac{\xi^2}{4a}\right)$ $(\xi>0)$
11	$\cos ax^2$ $(a>0)$	$\frac{1}{2\sqrt{a}}\left(\cos\frac{\xi^2}{4a}+\sin\frac{\xi^2}{4a}\right)$ $(\xi>0)$
12	$\frac{\sinh ax}{\sinh bx}$ $(\lvert\mathrm{Re}\ a\rvert<\mathrm{Re}\ b)$	$\sqrt{\frac{\pi}{2}}\frac{\sin\frac{a\pi}{b}}{b\left(\cosh\frac{\xi\pi}{b}+\cos\frac{a\pi}{b}\right)}$ $(\xi>0)$
13	$\frac{\cosh ax}{\cosh bx}$ $(\lvert\mathrm{Re}\ a\rvert<\mathrm{Re}\ b)$	$\sqrt{2\pi}\frac{\cos\frac{a\pi}{2b}\cosh\frac{\xi\pi}{2b}}{b\left(\cos\frac{a\pi}{b}+\cosh\frac{\xi\pi}{b}\right)}$ $(\xi>0)$
14	$\frac{\mathrm{J}_0(ax)}{x^2+b^2}$ $(a>0,\ \mathrm{Re}\ b>0)$	$\sqrt{\frac{\pi}{2}}\frac{\mathrm{I}_0(ab)}{b\mathrm{e}^{b\xi}}$ $(a<\xi<\infty)$
15	$\frac{x\mathrm{J}_0(ax)}{x^2+b^2}$ $(a>0,\ \mathrm{Re}\ b>0)$	$\sqrt{\frac{2}{\pi}}\cosh b\xi\,\mathrm{K}_0(ab)$ $(0<\xi<a)$

Ⅳ 附 录

Ⅳ.1 常用函数的定义和性质

Ⅳ.1.1 初等函数

Ⅳ.1.1.1 幂函数和代数函数

1. 幂函数

形如 $y = x^{\mu}$ 的函数称为幂函数，式中，μ 为任何实常数. 幂函数的定义域随不同的 μ 而异，但无论 μ 为何值，在 $(0, +\infty)$ 内幂函数总是有定义的.

2. 代数函数

代数函数包括有理函数（多项式与多项式之商）和无理函数（有理函数的根式）两类，代数函数是解析函数.

Ⅳ.1.1.2 指数函数和对数函数

1. 指数函数

定义 $y = e^{x}$ 为指数函数，其中，e 为自然对数的底，x 为指数，通常是实数. 指数函数满足加法定理

$$e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$$

当指数为复数 $z = x + iy$ 时，则称 $e^{z} = e^{x}(\cos y + i\sin y)$ 是复数 z 的指数函数，加法定理

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

依然成立. 由于 $e^{2\pi i}=1$,因此 e^z 是以 $2\pi i$ 为周期的周期函数.

2. 对数函数

(1) 定义

指数函数的反函数称为对数函数. 设 $z=e^w$,则 $w=\mathrm{Ln}z$ 为对数函数. 因此有

$$\mathrm{Ln}z = \ln|z| + i\arg z = \ln|z| + i(\arg z + 2k\pi) \quad (k = 0, \pm 1, \pm 2, \cdots)$$

$\mathrm{Ln}z$ 是一个无穷多值函数,其中

$$\ln z = \ln|z| + i\arg z \quad (-\pi < \arg z \leqslant \pi)$$

称为对数函数 $\mathrm{Ln}z$ 的主值,$\ln z$ 是单值函数. 所以

$$\mathrm{Ln}z = \ln z + 2k\pi i$$

(2) 性质

$$\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}z_1 + \mathrm{Ln}z_2$$

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \quad (-\pi < \arg z_1 + \arg z_2 < \pi)$$

$$\mathrm{Ln}\frac{z_1}{z_2} = \mathrm{Ln}z_1 - \mathrm{Ln}z_2$$

$$\ln\frac{z_1}{z_2} = \ln z_1 - \ln z_2 \quad (-\pi < \arg z_1 - \arg z_2 < \pi)$$

(3) 特殊值

$$\ln 0 = -\infty, \qquad \ln 1 = 0$$

$$\ln e = 1, \qquad \ln(-1) = i\pi$$

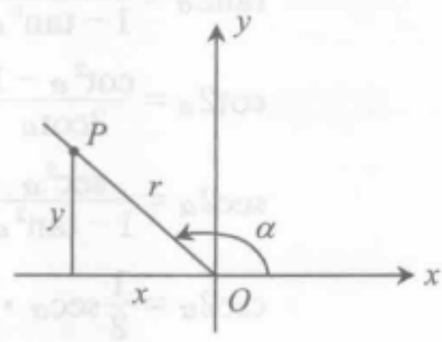
$$\ln(\pm i) = \pm\frac{i\pi}{2}$$

Ⅳ.1.1.3 三角函数和反三角函数

1. 三角函数

(1) 三角函数的定义

三角函数又称圆函数. 设任意角 α 的顶点为原点,始边位于 x 轴的正半轴,终边上任一点 P 的坐标为(x,y),P 点离原点的距离为 $r=\sqrt{x^2+y^2}$(如图所示),则任意角 α 的三角函数为



正弦函数

$$\sin\alpha = \frac{y}{r}$$

余弦函数

$$\cos\alpha = \frac{x}{r}$$

正切函数

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{y}{x}$$

余切函数

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{x}{y}$$

正割函数

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{r}{x}$$

余割函数

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{r}{y}$$

(2) 三角函数之间的关系

$$\sin\alpha \cdot \csc\alpha = 1$$
$$\cos\alpha \cdot \sec\alpha = 1$$
$$\tan\alpha \cdot \cot\alpha = 1$$
$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sec^2\alpha - \tan^2\alpha = 1$$
$$\csc^2\alpha - \cot^2\alpha = 1$$

(3) 和差公式

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$
$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

(4) 倍角公式

$$\sin2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

$$\cos2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$$\tan2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cot2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

$$\sec2\alpha = \frac{\sec^2\alpha}{1-\tan^2\alpha} = \frac{\cot\alpha + \tan\alpha}{\cot\alpha - \tan\alpha}$$

$$\csc2\alpha = \frac{1}{2}\sec\alpha \cdot \csc\alpha = \frac{1}{2}(\tan\alpha + \cot\alpha)$$

$$\sin3\alpha = -4\sin^3\alpha + 3\sin\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\cot 3\alpha = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$$

$$\sin n\alpha = n\cos^{n-1}\alpha\sin\alpha - C_n^3\cos^{n-3}\alpha\sin^3\alpha + C_n^5\cos^{n-5}\alpha\sin^5\alpha - \cdots \quad (n\text{ 为正整数})$$

$$\cos n\alpha = \cos^n\alpha - C_n^2\cos^{n-2}\alpha\sin^2\alpha + C_n^4\cos^{n-4}\alpha\sin^4\alpha - C_n^6\cos^{n-6}\alpha\sin^6\alpha + \cdots \quad (n\text{ 为正整数})$$

(5) 半角公式

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$$

$$\sec\frac{\alpha}{2} = \pm\sqrt{\frac{2\sec\alpha}{\sec\alpha+1}}$$

$$\csc\frac{\alpha}{2} = \pm\sqrt{\frac{2\sec\alpha}{\sec\alpha-1}}$$

(6) 和差化积公式

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$$

$$\cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha\pm\beta)}{\sin\alpha\sin\beta}$$

$$\tan\alpha \pm \cot\beta = \pm\frac{\cos(\alpha\mp\beta)}{\cos\alpha\sin\beta}$$

$$\sin\alpha + \cos\alpha = \sqrt{2}\sin\left(\frac{\pi}{4}+\alpha\right)$$

$$\sin\alpha - \cos\alpha = -\sqrt{2}\cos\left(\frac{\pi}{4} + \alpha\right)$$

$$\tan\alpha + \cot\alpha = 2\csc 2\alpha$$

$$\tan\alpha - \cot\alpha = -2\cot 2\alpha$$

$$\frac{1 + \tan\alpha}{1 - \tan\alpha} = \tan\left(\frac{\pi}{4} + \alpha\right)$$

$$\frac{1 + \cot\alpha}{1 - \cot\alpha} = -\cot\left(\frac{\pi}{4} - \alpha\right)$$

(7) 积化和差公式

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha + \beta) - \cos(\alpha - \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

(8) 棣莫弗(de Moivre)公式

$$(\cos\alpha + \mathrm{i}\sin\alpha)^n = \cos n\alpha + \mathrm{i}\sin n\alpha$$

(9) 欧拉(Euler)公式

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

$$\mathrm{e}^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta$$

2. 反三角函数

(1) 反三角函数的定义

反三角函数是三角函数的反函数,一般是多值函数. 它们分别是反正弦、反余弦、反正切、反余切、反正割和反余割函数. 若 $x = \sin y$,则 $y = \arcsin x$,我们把 $\arcsin x$ 叫做 x 的反正弦函数,其余类推. 通常把反三角函数的值限制在一定的范围内,称为主值,并分别记为

反正弦函数

$$y = \arcsin x \quad \left(\text{主值范围为}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

反余弦函数

$$y = \arccos x \quad (\text{主值范围为}[0, \pi])$$

反正切函数

$$y = \arctan x \quad \left(\text{主值范围为}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

反余切函数

$$y = \operatorname{arccot} x \quad (\text{主值范围为}(0, \pi))$$

反正割函数

$$y = \operatorname{arcsec} x \quad \left(\text{主值范围为}\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]\right)$$

反余割函数

$$y = \operatorname{arccsc} x \quad \left(\text{主值范围为}\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]\right)$$

有时反三角函数的主值也记为 $y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$, $y = \cot^{-1} x$, $y = \sec^{-1} x$, $y = \csc^{-1} x$,但本书不用.

(2) 反三角函数的性质

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

(3) 反三角函数之间的关系

$$\arcsin x = \arccos \sqrt{1 - x^2} = \arctan \frac{x}{\sqrt{1 - x^2}} = \operatorname{arcsec} \frac{1}{\sqrt{1 - x^2}} = \operatorname{arccsc} \frac{1}{x}$$

$$\arccos x = \arcsin \sqrt{1 - x^2} = \arctan \frac{\sqrt{1 - x^2}}{x} = \operatorname{arccsc} \frac{1}{\sqrt{1 - x^2}} = \operatorname{arcsec} \frac{1}{x}$$

$$\arctan x = \arcsin \frac{x}{\sqrt{1 + x^2}} = \arccos \frac{1}{\sqrt{1 + x^2}} = \operatorname{arcsec} \sqrt{1 + x^2} = \operatorname{arccot} \frac{1}{x}$$

$$\operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1 + x^2}} = \arccos \frac{x}{\sqrt{1 + x^2}} = \operatorname{arccsc} \sqrt{1 + x^2} = \arctan \frac{1}{x}$$

$$\operatorname{arcsec} x = \arcsin \frac{\sqrt{x^2 - 1}}{x} = \arctan \sqrt{x^2 - 1} = \operatorname{arccsc} \frac{x}{\sqrt{x^2 - 1}} = \arccos \frac{1}{x}$$

$$\operatorname{arccsc} x = \arccos \frac{\sqrt{x^2 - 1}}{x} = \arctan \frac{1}{\sqrt{x^2 - 1}} = \operatorname{arcsec} \frac{x}{\sqrt{x^2 - 1}} = \arcsin \frac{1}{x}$$

Ⅳ.1.1.4 双曲函数和反双曲函数

1. 双曲函数

(1) 双曲函数的定义

双曲正弦函数

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

双曲余弦函数

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

双曲正切函数

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

双曲余切函数

$$\coth x=\frac{\cosh x}{\sinh x}=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

双曲正割函数

$$\operatorname{sech} x=\frac{1}{\cosh x}=\frac{2}{e^{x}+e^{-x}}$$

双曲余割函数

$$\operatorname{csch} x=\frac{1}{\sinh x}=\frac{2}{e^{x}-e^{-x}}$$

(2) 双曲函数之间的关系

$$\cosh^2 x-\sinh^2 x=1$$
$$\tanh^2 x+\operatorname{sech}^2 x=1$$
$$\coth^2 x-\operatorname{csch}^2 x=1$$

(3) 和差的双曲函数

$$\sinh(x\pm y)=\sinh x\cosh y\pm\cosh x\sinh y$$
$$\cosh(x\pm y)=\cosh x\cosh y\pm\sinh x\sinh y$$
$$\tanh(x\pm y)=\frac{\tanh x\pm\tanh y}{1\pm\tanh x\tanh y}$$
$$\coth(x\pm y)=\frac{1\pm\coth x\coth y}{\coth x\pm\coth y}$$

(4) 双曲函数的和差

$$\sinh x\pm\sinh y=2\sinh\frac{x\pm y}{2}\cosh\frac{x\mp y}{2}$$
$$\cosh x+\cosh y=2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$$
$$\cosh x-\cosh y=2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$$
$$\tanh x\pm\tanh y=\frac{\sinh(x\pm y)}{\cosh x\cosh y}$$
$$\coth x\pm\coth y=\pm\frac{\sinh(x\pm y)}{\sinh x\sinh y}$$

(5) 倍角公式

$$\sinh 2x=2\sinh x\cosh x=\frac{2\tanh x}{1-\tanh^2 x}$$
$$\cosh 2x=\sinh^2 x+\cosh^2 x=1+2\sinh^2 x$$
$$=2\cosh^2 x-1=\frac{1+\tanh^2 x}{1-\tanh^2 x}$$

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

$$\coth 2x = \frac{1+\coth^2 x}{2\coth x}$$

(6) 半角公式

$$\sinh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}} \quad (x>0,\text{取正号};x<0,\text{取负号})$$

$$\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh\frac{x}{2} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}$$

$$\coth\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{\cosh x - 1}} = \frac{\sinh x}{\cosh x - 1} = \frac{\cosh x + 1}{\sinh x}$$

(7) 双曲函数的棣莫弗(de Moivre)公式

$$(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx \quad (n\text{ 为正整数})$$

(8) 双曲函数与三角函数的关系

$$\sinh z = -\mathrm{i}\sin \mathrm{i}z, \qquad \sin z = -\mathrm{i}\sinh \mathrm{i}z$$

$$\cosh z = \cos \mathrm{i}z, \qquad \cos z = \cosh \mathrm{i}z$$

$$\tanh z = -\mathrm{i}\tan \mathrm{i}z, \qquad \tan z = -\mathrm{i}\tanh \mathrm{i}z$$

$$\coth z = \mathrm{i}\cot \mathrm{i}z, \qquad \cot z = \mathrm{i}\coth \mathrm{i}z$$

在上述诸式中，$\mathrm{i}=\sqrt{-1}$.

2. 反双曲函数

(1) 反双曲函数的定义

若 $x=\sinh y$，则 $y=\operatorname{arsinh} x$ 称为反双曲正弦函数，x 的定义域为$(-\infty,+\infty)$.

若 $x=\cosh y$，则 $y=\operatorname{arcosh} x$ 称为反双曲余弦函数，x 的定义域为$[1,+\infty)$.

若 $x=\tanh y$，则 $y=\operatorname{artanh} x$ 称为反双曲正切函数，x 的定义域为$(-1,1)$.

若 $x=\coth y$，则 $y=\operatorname{arcoth} x$ 称为反双曲余切函数，x 的定义域为$(-\infty,-1)\cup(1,+\infty)$.

若 $x=\operatorname{sech} y$，则 $y=\operatorname{arsech} x$ 称为反双曲正割函数，x 的定义域为$(0,1]$.

若 $x=\operatorname{csch} y$，则 $y=\operatorname{arcsch} x$ 称为反双曲余割函数，x 的定义域为$(-\infty,0)\cup(0,+\infty)$.

反双曲正弦、余弦、正切、余切、正割、余割函数有时也分别记为 $\sinh^{-1}x$，$\cosh^{-1}x$，$\tanh^{-1}x$，$\coth^{-1}x$，$\operatorname{sech}^{-1}x$，$\operatorname{csch}^{-1}x$，但本书不用.

(2) 反双曲函数的相互关系

$$\operatorname{arsinh} x = \pm\operatorname{arcosh}\sqrt{x^2+1} = \operatorname{artanh}\frac{x}{\sqrt{x^2+1}} = \operatorname{arcoth}\frac{\sqrt{x^2+1}}{x}$$

$$\operatorname{arcosh} x = \pm \operatorname{arsinh} \sqrt{x^2-1} = \pm \operatorname{artanh} \frac{\sqrt{x^2-1}}{x} = \pm \operatorname{arcoth} \frac{x}{\sqrt{x^2-1}}$$

$$\operatorname{artanh} x = \operatorname{arsinh} \frac{x}{\sqrt{1-x^2}} = \pm \operatorname{arcosh} \frac{1}{\sqrt{1-x^2}} = \operatorname{arcoth} \frac{1}{x}$$

$$\operatorname{arcoth} x = \operatorname{arsinh} \frac{1}{\sqrt{x^2-1}} = \pm \operatorname{arcosh} \frac{x}{\sqrt{x^2-1}} = \operatorname{artanh} \frac{1}{x}$$

$$\operatorname{arsech} x = \operatorname{arsinh} \frac{\sqrt{1-x^2}}{x} = \operatorname{arcosh} \frac{1}{x} = \operatorname{artanh} \sqrt{1-x^2}$$

$$\operatorname{arcsch} x = \operatorname{arsinh} \frac{1}{x} = \operatorname{arcosh} \frac{\sqrt{x^2+1}}{x} = \operatorname{artanh} \frac{1}{\sqrt{x^2+1}}$$

在上述诸式中，当 $x>0$ 时，取正号；当 $x<0$ 时，取负号.

(3) 基本公式

$$\operatorname{arsinh} x \pm \operatorname{arsinh} y = \operatorname{arsinh}\left(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\right)$$

$$\operatorname{arcosh} x \pm \operatorname{arcosh} y = \operatorname{arcosh}\left(xy \pm \sqrt{(x^2-1)(y^2-1)}\right)$$

$$\operatorname{artanh} x \pm \operatorname{artanh} y = \operatorname{artanh} \frac{x \pm y}{1 \pm xy}$$

(4) 双曲函数与对数函数的关系

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \quad (x > 0)$$

$$\operatorname{arcosh} x = \pm \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geqslant 1)$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x} \quad (|x| < 1)$$

$$\operatorname{arcoth} x = \frac{1}{2}\ln\frac{x+1}{x-1} \quad (|x| > 1)$$

$$\operatorname{arsech} x = \pm \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right) \quad (0 < x \leqslant 1)$$

$$\operatorname{arcsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{x}\right) \quad (x > 0)$$

Ⅳ.1.2 特殊函数

Ⅳ.1.2.1 Γ 函数(第二类欧拉积分)

1. 定义

Γ 函数(Gamma Function,译称伽马函数)的定义为

$$\Gamma(z)=\int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\mathrm{d}t \quad (\mathrm{Re}\ z>0)$$

上式的右边称为第二类欧拉积分.

2. 围道积分

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi\mathrm{i}}\int_{(-\infty)}^{(0_+)} t^{-z}\mathrm{e}^{t}\mathrm{d}t \quad (|\arg t|<\pi)$$

其中的围道从负实轴无穷远处($t=-\infty$)出发,正向绕原点一周,再回到出发点.

3. 无穷乘积形式

欧拉无穷乘积形式为

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}} \quad (z\neq 0,-1,-2,\cdots,-n)$$

该式对于任何 z,除了极点 $z=-n$ 外都是成立的,因此它是普遍的 $\Gamma(z)$ 的定义.

魏尔斯特拉斯(Weierstrass)无穷乘积形式为

$$\frac{1}{\Gamma(z)}=z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\mathrm{e}^{-\frac{z}{n}}\right]$$

这里,γ 为欧拉常数.

4. 递推公式及有关公式

$$\Gamma(1+z)=z\Gamma(z) \quad (\mathrm{Re}\ z>0)$$

$$\Gamma(1-z)=-z\Gamma(-z)$$

$$\Gamma(z)\Gamma(-z)=-\frac{\pi}{z\sin\pi z} \quad (z\text{ 为非整数})$$

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \quad (z\text{ 为非整数})$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos\pi z}$$

$$\Gamma(n+z)\Gamma(n-z)$$

$$=\frac{\pi z}{\sin\pi z}[(n-1)!]^2\prod_{k=1}^{n-1}\left(1-\frac{z^2}{k^2}\right) \quad (n=1,2,3,\cdots)$$

$$\Gamma\left(n+\frac{1}{2}+z\right)\Gamma\left(n+\frac{1}{2}-z\right)$$

$$=\frac{1}{\cos\pi z}\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2\prod_{k=1}^{n}\left[1-\frac{4z^2}{(2k-1)^2}\right]\quad(n=1,2,3,\cdots)$$

$$\Gamma(nz)=(2\pi)^{\frac{1}{2}(1-n)}n^{nz-\frac{1}{2}}\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right)\quad(n=1,2,3,\cdots)$$

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

$$\int_0^{\frac{\pi}{2}}\cos^{2x-1}\theta\sin^{2y-1}\theta\mathrm{d}\theta=\frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}\quad(x>0,\ y>0)$$

5. 斯特林(Stirling)公式

$$\Gamma(z)=\sqrt{2\pi}z^{z-\frac{1}{2}}\mathrm{e}^{-z}\left[1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}\right.$$

$$\left.+\frac{163879}{209018880z^5}+\cdots\right]\quad(|\arg z|<\pi,\ |z|\to\infty)$$

$$n!=\sqrt{2n\pi}\left(\frac{n}{\mathrm{e}}\right)^n\mathrm{e}^{\frac{\theta_n}{12n}}\quad(0<\theta_n<1)$$

6. 特殊值

$$\Gamma(n+1)=n!\quad(n=0,1,2,\cdots;\ 0!=1)$$

$$\Gamma(1)=1,\qquad\Gamma(2)=1,\qquad\Gamma(3)=2!,\qquad\Gamma(4)=3!$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}=1.7724538509\cdots=\left(-\frac{1}{2}\right)!$$

$$\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}=0.8862269254\cdots=\left(\frac{1}{2}\right)!$$

$$\Gamma\left(n+\frac{1}{4}\right)=\frac{1\cdot5\cdot9\cdot13\cdot\cdots\cdot(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right)\quad(n=1,2,3,\cdots)$$

$$\Gamma\left(\frac{1}{4}\right)=3.6256099082\cdots$$

$$\Gamma\left(n+\frac{1}{3}\right)=\frac{1\cdot4\cdot7\cdot10\cdot\cdots\cdot(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right)\quad(n=1,2,3,\cdots)$$

$$\Gamma\left(\frac{1}{3}\right)=2.6789385347\cdots$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\cdot3\cdot5\cdot7\cdot\cdots\cdot(2n-1)}{2^n}\Gamma\left(\frac{1}{2}\right)\quad(n=1,2,3,\cdots)$$

$$\Gamma\left(n+\frac{2}{3}\right)=\frac{2\cdot5\cdot8\cdot11\cdot\cdots\cdot(3n-1)}{3^n}\Gamma\left(\frac{2}{3}\right)\quad(n=1,2,3,\cdots)$$

$$\Gamma\left(\frac{2}{3}\right)=1.3541179394\cdots$$

$$\Gamma\left(n+\frac{3}{4}\right)=\frac{3\cdot7\cdot11\cdot15\cdot\cdots\cdot(4n-1)}{4^n}\Gamma\left(\frac{3}{4}\right)\quad(n=1,2,3,\cdots)$$

$$\Gamma\left(\frac{3}{4}\right) = 1.2254167024\cdots$$

Ⅳ.1.2.2 B 函数(第一类欧拉积分)

1. 定义

B 函数(Beta Function)为双变量函数,译称贝塔函数,其定义为

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t \quad (\mathrm{Re}\ x > 0,\ \mathrm{Re}\ y > 0)$$

上式的右边称为**第一类欧拉积分**. 如果在上式中作变量替换$\left(令\ t = \frac{1}{1+z}, 或\ t = \sin^2\theta\right)$,可得到不同的表达式:

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t \quad (\mathrm{Re}\ x > 0,\ \mathrm{Re}\ y > 0)$$

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \mathrm{d}\theta \quad (\mathrm{Re}\ x > 0,\ \mathrm{Re}\ y > 0)$$

2. B 函数的对称特性

$$B(x,y) = B(y,x)$$

3. B 函数与 Γ 函数的关系

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Ⅳ.1.2.3 ψ 函数

1. ψ 函数的定义及其他表达式

ψ 函数(Psi Function)的定义为

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

ψ 函数还有两种不同的表达式:

$$\psi(z) = \int_0^\infty \left(\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}\right)\mathrm{d}t \quad (\mathrm{Re}\ z > 0) \quad (高斯积分)$$

$$\psi(z) = \int_0^\infty \left[\frac{\mathrm{e}^{-t}}{t} - \frac{1}{t(1+t)^z}\right]\mathrm{d}t \quad (\mathrm{Re}\ z > 0) \quad (狄利克雷积分)$$

2. ψ 函数的有关公式

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+x}\right) \quad (\gamma\ 为欧拉常数)$$

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

$$\psi(1-x)-\psi(x)=\pi\cot\pi x$$

$$\psi(x)+\psi\left(x+\frac{1}{2}\right)-2\ln 2=2\psi(2x)$$

3. ψ 函数的特殊值

$$\psi(1)=-\gamma \quad (\gamma\text{ 为欧拉常数,以下同})$$

$$\psi\left(\frac{1}{2}\right)=-\gamma-2\ln 2$$

$$\psi\left(\frac{1}{3}\right)=-\gamma-2\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)}$$

$$\psi(n+1)=-\gamma+\sum_{k=1}^{n}\frac{1}{k} \quad (n=1,2,3,\cdots)$$

$$\psi(x+1)=-\gamma+\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)x^{n}$$

$$\left[-1<x<1,\ \zeta(n+1)=\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}\right]$$

$$\psi^{(m)}(x)=\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}[\ln\Gamma(x)] \quad (m=1,2,3,\cdots)$$

$$\psi'(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^{2}} \quad (x>0)$$

$$\psi'(1)=\frac{\pi^{2}}{6}=1.644934066\cdots$$

$$\psi'\left(\frac{1}{2}\right)=\frac{\pi^{2}}{2}=4.934802200\cdots$$

$$\psi'\left(\frac{1}{3}\right)=\sum_{n=0}^{\infty}\frac{1}{\left(\frac{1}{3}+n\right)^{2}}=10.09559712\cdots$$

$$\psi'(n)=\frac{\pi^{2}}{6}-\sum_{k=1}^{n-1}\frac{1}{k^{2}} \quad (n=1,2,3,\cdots)$$

$$\psi'(-n)=\infty \quad (n=1,2,3,\cdots)$$

4. ψ 函数的渐近表达式

$$\psi(x+1)\approx\ln x+\frac{1}{2x}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{B_{2n}}{n}x^{-2n} \quad (x\to\infty,\ B_{2n}\text{ 为伯努利数})$$

Ⅳ.1.2.4 误差函数 erf(x)和补余误差函数 erfc(x)

1. 误差函数(概率积分)的定义

误差函数的定义为

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\mathrm{d}t = \Phi(x)$$

这里定义的误差函数 erf(x)是与概率积分 $\Phi(x)$相等的,此处的概率函数有别于以坐标原点为对称点的正态分布的概率函数.

2. 补余误差函数(补余概率积分)的定义

补余误差函数的定义为

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}\mathrm{d}t$$

3. 特性与关系式

$$\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$$

$$\mathrm{erf}(-x) = -\mathrm{erf}(x)$$

$$\mathrm{erfc}(-x) = 2 - \mathrm{erfc}(x)$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)n!} = \frac{2}{\sqrt{\pi}}\left(x - \frac{1}{1!}\frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{5} - \frac{1}{3!}\frac{x^7}{7} + \cdots\right)$$

$$(-\infty < x < \infty)$$

$$\mathrm{erfc}(x) \approx \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{(2x^2)^n}\right] \quad (x \to \infty)$$

4. 特殊值

$$\mathrm{erf}(0) = 0, \qquad \mathrm{erf}(\infty) = 1, \qquad \mathrm{erf}(-\infty) = -1$$

$$\mathrm{erfc}(-\infty) = 2, \qquad \mathrm{erfc}(\infty) = 0$$

$$\mathrm{erf}(x_0) = \mathrm{erfc}(x_0) = \frac{1}{2} \quad (\text{当 } x_0 = 0.476936\cdots\text{时})$$

5. 误差函数与标准正态概率函数的关系

$$\int_0^x f(t)\mathrm{d}t = \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

这里,$f(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}}$为标准正态分布的概率密度函数.

6. 误差函数与等离子体弥散函数 w(z)的关系

$$\mathrm{w}(z) = \mathrm{e}^{-z^2}\mathrm{erfc}(-\mathrm{i}z) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-t^2}}{t-z}\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(\mathrm{i}z)^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

7. 误差函数与道生(Dawson)积分 F(x)的关系

$$F(x) = e^{-x^2}\int_0^x e^{t^2}dt = -\frac{1}{2}i\sqrt{\pi}e^{-x^2}\operatorname{erf}(ix)$$

Ⅳ.1.2.5 菲涅耳(Fresnel)函数 S(z)和 C(z)

菲涅耳函数定义为

$$S(z) = \int_0^z \sin\frac{\pi t^2}{2}dt$$

$$C(z) = \int_0^z \cos\frac{\pi t^2}{2}dt$$

它们都是 z 的整函数,也可用级数表述:

$$S(z) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{\pi}{2}\right)^{2k+1}\frac{z^{4k+3}}{(2k+1)!(4k+3)} \quad (|z|<\infty)$$

$$C(z) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{\pi}{2}\right)^{2k}\frac{z^{4k+1}}{(2k)!(4k+1)} \quad (|z|<\infty)$$

Ⅳ.1.2.6 正弦积分 Si(z),si(z)和余弦积分 Ci(z),ci(z)

1. 正弦积分

$$Si(z) = \int_0^z \frac{\sin t}{t}dt = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!(2k+1)} \quad (|z|<\infty)$$

$$si(z) = -\int_z^{\infty}\frac{\sin t}{t}dt = Si(z) - \frac{\pi}{2}$$

2. 余弦积分

$$Ci(z) = ci(z) = -\int_z^{\infty}\frac{\cos t}{t}dt = \ln z + \sum_{k=1}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!2k} + \gamma$$

($|\arg z|<\pi$, γ 为欧拉常数)

3. 双曲正弦积分与双曲余弦积分

$$shi(z) = \int_0^z \frac{\sinh t}{t}dt = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!(2k+1)}$$

$$chi(z) = \int_0^z \frac{\cosh t - 1}{t}dt + \ln z + \gamma = \sum_{k=1}^{\infty}\frac{z^{2k}}{(2k)!2k} + \ln z + \gamma$$

(γ 为欧拉常数)

Ⅳ.1.2.7 指数积分 Ei(z)和对数积分 li(z)

1. 指数积分

(1) 指数积分的定义

指数积分的定义为

$$\mathrm{Ei}(z)=\int_{-\infty}^{z}\frac{\mathrm{e}^{t}}{t}\mathrm{d}t=\ln(-z)+\sum_{k=1}^{\infty}\frac{z^{k}}{k!k}+\gamma\quad(\gamma\text{ 为欧拉常数})$$

上式在除去半实轴$(0,+\infty)$的 z 平面内单值解析.

$$\overline{\mathrm{Ei}}(x)=-(\mathrm{P.V.})\int_{-x}^{\infty}\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t$$

$$=-\lim_{\varepsilon\to+0}\left(\int_{-x}^{-\varepsilon}\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t+\int_{\varepsilon}^{\infty}\frac{\mathrm{e}^{-t}}{t}\mathrm{d}t\right)\quad(0<x<\infty)$$

$$\mathrm{E}_n(x)=\int_{1}^{\infty}\frac{\mathrm{e}^{-xt}}{t^{n}}\mathrm{d}t\quad(0<x<\infty)$$

(2) 特殊值

$$\mathrm{Ei}(x)=\gamma+\ln x+\sum_{n=1}^{\infty}\frac{x^{n}}{n!n}\quad(x>0)$$

$$\mathrm{E}_1(x)=-\gamma-\ln x-\sum_{n=1}^{\infty}\frac{(-1)^{n}x^{n}}{n!n}\quad(x>0)$$

(3) 指数积分的渐近表达式

$$\mathrm{Ei}(x)\approx\frac{\mathrm{e}^{x}}{x}\sum_{n=0}^{\infty}\frac{n!}{x^{n}}\quad(x>0,\ x\to\infty)$$

$$\mathrm{E}_1(x)\approx\frac{\mathrm{e}^{-x}}{x}\sum_{n=0}^{\infty}\frac{(-1)^{n}n!}{x^{n}}\quad(x>0,\ x\to\infty)$$

2. 对数积分

对数积分的定义为

$$\mathrm{li}(z)=\int_{0}^{z}\frac{\mathrm{d}t}{\ln t}=\mathrm{Ei}(\ln z)$$

它在除去$(-\infty,0)$和$[1,+\infty)$的 z 平面内单值解析.

当 z 为正实数 x 时,积分主值为

$$\overline{\mathrm{li}}(x)=(\mathrm{P.V.})\int_{0}^{x}\frac{\mathrm{d}t}{\ln t}=\lim_{\varepsilon\to0}\left(\int_{0}^{1-\varepsilon}\frac{\mathrm{d}t}{\ln t}+\int_{1+\varepsilon}^{x}\frac{\mathrm{d}t}{\ln t}\right)$$

$$=\overline{\mathrm{Ei}}(\ln x)\quad(1<x<\infty)$$

这里,$\overline{\mathrm{Ei}}$为指数积分.

Ⅳ.1.2.8 勒让德(Legendre)椭圆积分 $F(k,\varphi)$,$E(k,\varphi)$,$\Pi(h,k,\varphi)$

1. 第一类椭圆积分

$$F(k,\varphi)=\int_0^{\sin\varphi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\quad(k^2<1)$$

2. 第二类椭圆积分

$$E(k,\varphi)=\int_0^{\sin\varphi}\sqrt{\frac{1-k^2t^2}{1-t^2}}dt=\int_0^{\varphi}\sqrt{1-k^2\sin^2\theta}d\theta\quad(k^2<1)$$

3. 第三类椭圆积分

$$\Pi(h,k,\varphi)=\int_0^{\sin\varphi}\frac{dt}{(1+ht^2)\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$=\int_0^{\varphi}\frac{d\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}\quad(k^2<1,\ h\text{ 为非负整数})$$

这里,k 称为这些积分的模数,$k'=\sqrt{1-k^2}$称为补模数,h 称为第三类椭圆积分的参数.

Ⅳ.1.2.9 完全椭圆积分 $K(k)$,$E(k)$,$\Pi(h,k)$

1. 第一类完全椭圆积分

$$K=K(k)=F\left(k,\frac{\pi}{2}\right)=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\quad(k^2<1)$$

2. 第二类完全椭圆积分

$$E=E(k)=E\left(k,\frac{\pi}{2}\right)=\int_0^1\sqrt{\frac{1-k^2t^2}{1-t^2}}dt$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}d\theta\quad(k^2<1)$$

3. 第三类完全椭圆积分

$$\Pi(h,k)=\Pi\left(h,k,\frac{\pi}{2}\right)=\int_0^1\frac{dt}{(1+ht^2)\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}\quad(k^2<1)$$

并定义

$$K'(k) = K(k'), \quad E'(k) = E(k')$$
$$K'(k') = K(k), \quad E'(k') = E(k)$$

第一类和第二类完全椭圆积分也可写成级数形式：

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} k^{2n}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} k^{2n}$$

Ⅳ.1.2.10 贝塞尔(Bessel)函数(柱函数)$J_\nu(z)$，$N_\nu(z)$，$H_\nu^{(1)}(z)$，$H_\nu^{(2)}(z)$，$I_\nu(z)$，$K_\nu(z)$

1. 贝塞尔方程

贝塞尔方程为

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)y = 0$$

其中，ν 是常数，称为方程的阶或方程的解的阶，它可以是任何实数或复数.

贝塞尔方程的解为**贝塞尔函数**，或称柱函数：

$$J_{\pm\nu}(z) = \left(\frac{z}{2}\right)^{\pm\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\Gamma(\pm\nu + k + 1)}\left(\frac{z}{2}\right)^{2k}$$

（ν 为常数，$|\arg z| < \pi$）

它们是 ν 阶贝塞尔方程的两个解，$J_{\pm\nu}(z)$称为**第一类贝塞尔函数**. 除 ν 为整数的情形外，$J_\nu(z)$和 $J_{-\nu}(z)$线性无关；当 ν 为整数时，$J_\nu(z)$和 $J_{-\nu}(z)$线性相关，此时可构造方程的另一个解，称为**诺伊曼**(Neumann)**函数**：

$$N_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$$

$N_\nu(z)$称为**第二类贝塞尔函数**. $J_\nu(z)$和 $N_\nu(z)$也是 ν 阶贝塞尔方程的两个线性独立解.

由 $J_\nu(z)$和 $N_\nu(z)$可线性组合成**汉克尔**(Hankel)**函数**，其定义为

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z)$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z)$$

$H_\nu^{(1)}(z)$和 $H_\nu^{(2)}(z)$也是 ν 阶贝塞尔方程的两个线性独立解，分别叫做第一种汉克尔函数和第二种汉克尔函数，也称为**第三类贝塞尔函数**.

2. 虚自变量贝塞尔方程

在方程

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

中，令 $z=\mathrm{i}x$，此处 x 为实数，则方程变为

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}y}{\mathrm{d}z}-\left(1+\frac{\nu^2}{z^2}\right)y=0$$

该方程称为**虚自变量贝塞尔方程**.

它的第一个解为**第一类虚自变量贝塞尔函数**

$$\mathrm{I}_\nu(z)=\begin{cases}\mathrm{e}^{-\mathrm{i}\frac{\nu\pi}{2}}\mathrm{J}_\nu\left(z\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) & \left(-\pi<\arg z\leqslant\frac{\pi}{2}\right)\\ \mathrm{e}^{\mathrm{i}\frac{3\nu\pi}{2}}\mathrm{J}_\nu\left(z\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}}\right) & \left(\frac{\pi}{2}<\arg z<\pi\right)\end{cases}$$

它的另一个线性独立解为**第二类虚自变量贝塞尔函数**

$$\mathrm{K}_\nu(z)=\frac{\pi}{2\sin\nu\pi}\left[\mathrm{I}_{-\nu}(z)-\mathrm{I}_\nu(z)\right]$$

或

$$\mathrm{K}_\nu(z)=\frac{\mathrm{i}\pi}{2}\mathrm{e}^{\mathrm{i}\frac{\nu\pi}{2}}\mathrm{H}_\nu^{(1)}\left(z\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)=-\mathrm{i}\frac{\pi}{2}\mathrm{e}^{-\mathrm{i}\frac{\nu\pi}{2}}\mathrm{H}_\nu^{(2)}\left(z\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\right)$$

3. 贝塞尔函数的特性及有关公式

(1) 第一类贝塞尔函数 $\mathrm{J}_\nu(z)$

定义：

$$\mathrm{J}_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}$$

关系式：

$$\mathrm{J}_{-n}(z)=(-1)^n\mathrm{J}_n(z)\quad(n=0,1,2,\cdots)$$

特殊值：

$$\mathrm{J}_0(0)=1,\quad \mathrm{J}_\nu(0)=0\quad(\nu>0)$$

递推关系：

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^\nu\mathrm{J}_\nu(z)\right]=z^\nu\mathrm{J}_{\nu-1}(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^{-\nu}\mathrm{J}_\nu(z)\right]=-z^{-\nu}\mathrm{J}_{\nu+1}(z)$$

$$\mathrm{J}_{\nu-1}(z)+\mathrm{J}_{\nu+1}(z)=\frac{2\nu}{z}\mathrm{J}_\nu(z)$$

$$\mathrm{J}_{\nu-1}(z)-\mathrm{J}_{\nu+1}(z)=2\mathrm{J}'_\nu(z)$$

积分表达式：

$$\mathrm{J}_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\mathrm{e}^{\mathrm{i}zt}(1-t^2)^{\nu-\frac{1}{2}}\mathrm{d}t\quad\left(\mathrm{Re}\,\nu>-\frac{1}{2}\right)$$

$$\mathrm{J}_n(z)=\frac{1}{\pi}\int_0^\pi\cos(n\theta-z\sin\theta)\mathrm{d}\theta\quad(n=0,1,2,\cdots)$$

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta)\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi}\cos(z\cos\theta)\mathrm{d}\theta$$

$$J_0(x) = \frac{2}{\pi}\int_0^{\infty}\sin(x\cosh t)\mathrm{d}t \quad (x > 0)$$

渐近公式：

$$J_\nu(x) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \quad (\nu \neq -1, -2, -3, \cdots; x \to 0^+)$$

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left[x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] \quad (x \to \infty)$$

半奇数阶贝塞尔函数：

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$$

$$J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\cos x + \frac{3}{x}\sin x\right]$$

(2) 第二类贝塞尔函数[诺伊曼(Neumann)函数]$N_\nu(z)$

定义：

$$N_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$$

关系式：

$$N_{-n}(z) = (-1)^n N_n(z) \quad (n = 0, 1, 2, \cdots)$$

特殊值：

$$N_n(0) = N_0(0) = -\infty$$

递推关系：

$$\frac{\mathrm{d}}{\mathrm{d}z}[z^\nu N_\nu(z)] = z^\nu N_{\nu-1}(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}[z^{-\nu} N_\nu(z)] = -z^{-\nu} N_{\nu+1}(z)$$

$$N_{\nu-1}(z) + N_{\nu+1}(z) = \frac{2\nu}{z}N_\nu(z)$$

$$N_{\nu-1}(z) - N_{\nu+1}(z) = 2N'_\nu(z)$$

积分表达式：

$$N_\nu(x) = -\frac{2}{\pi}\int_0^\infty \cos\left(x\cosh t - \frac{\nu\pi}{2}\right)\cosh\nu t\,dt$$

$$(-1 < \mathrm{Re}\,\nu < 1,\ x > 0)$$

$$N_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - \nu\theta)d\theta - \frac{1}{\pi}\int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\nu\pi)e^{-z\sinh t}dt$$

$$\left(\mathrm{Re}\,z > 0,\ |\arg z| < \frac{\pi}{2}\right)$$

$$N_0(z) = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \cos(z\cos\theta)[\ln(2z\sin^2\theta) + \gamma]d\theta$$

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)dt \quad (x > 0)$$

近似公式：

$$N_0(x) \approx \frac{2}{\pi}\ln x \quad (x \to 0^+)$$

$$N_\nu(x) \approx -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu \quad (\nu > 0,\ x \to 0^+)$$

$$N_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left[x - \frac{1}{2}\left(\nu + \frac{1}{2}\right)\pi\right] \quad (x \to \infty)$$

半奇数阶诺伊曼函数：

$$N_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\cos x$$

$$N_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$N_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

$$N_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} + \cos x\right)$$

$$N_{\frac{5}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\cos x + \frac{3}{x}\sin x\right]$$

$$N_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$$

(3) 第三类贝塞尔函数[汉克尔(Hankel)函数]$H_\nu^{(1)}(z)$和$H_\nu^{(2)}(z)$

定义：

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,N_\nu(z)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i\,N_\nu(z)$$

积分表达式：

$$H_\nu^{(1)}(x) = \frac{2}{i\pi}e^{-i\frac{\nu\pi}{2}}\int_0^\infty e^{ix\cosh t}\cosh\nu t\,dt \quad (-1 < \mathrm{Re}\,\nu < 1,\ x > 0)$$

$$H_\nu^{(2)}(x) = -\frac{2}{i\pi}e^{i\frac{\nu\pi}{2}}\int_0^\infty e^{-ix\cosh t}\cosh\nu t\,dt \quad (-1 < \mathrm{Re}\,\nu < 1,\ x > 0)$$

$$H_\nu^{(1)}(z) = -\frac{2^{\nu+1}iz^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}}\frac{\cos^{\nu-\frac{1}{2}}t\cdot e^{i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1}t}\exp(-2z\cot t)\,dt$$

$$\left(\mathrm{Re}\,\nu > -\frac{1}{2},\ \mathrm{Re}\,z > 0\right)$$

$$H_\nu^{(2)}(z) = -\frac{2^{\nu+1}iz^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}}\frac{\cos^{\nu-\frac{1}{2}}t\cdot e^{-i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1}t}\exp(-2z\cot t)\,dt$$

$$\left(\mathrm{Re}\,\nu > -\frac{1}{2},\ \mathrm{Re}\,z > 0\right)$$

$$H_0^{(1)}(x) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\exp\left(i\sqrt{x^2+t^2}\right)}{\sqrt{x^2+t^2}}dt \quad (x > 0)$$

(4) 第一类修正贝塞尔函数 $I_\nu(z)$

定义：

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{\nu+2k} \quad (|z| < \infty,\ |\arg z| < \pi)$$

$$I_\nu(z) = \begin{cases} e^{-i\frac{\nu\pi}{2}}J_\nu\left(ze^{i\frac{\pi}{2}}\right) & \left(-\pi < \arg z \leqslant \frac{\pi}{2}\right) \\ e^{i\frac{3\nu\pi}{2}}J_\nu\left(ze^{-i\frac{3\pi}{2}}\right) & \left(\frac{\pi}{2} < \arg z < \pi\right) \end{cases}$$

积分表达式：

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}e^{-zt}(1-t^2)^{\nu-\frac{1}{2}}dt \quad \left(\mathrm{Re}\,\nu > -\frac{1}{2}\right)$$

$$= \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\pi}\cosh(z\cos\theta)\sin^{2\nu}\theta\,d\theta \quad \left(\mathrm{Re}\,\nu > -\frac{1}{2}\right)$$

$$I_\nu(z) = \frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\cos\nu\theta\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^{\infty}e^{-z\cosh t-\nu t}dt$$

$$\left(\mathrm{Re}\,\nu > 0,\ |\arg z| \leqslant \frac{\pi}{2}\right)$$

关系式：

$$I_{-n}(z) = I_n(z) \quad (n = 0,1,2,\cdots)$$

特殊值：

$$I_0(0) = 1,\quad I_\nu(0) = 0 \quad (\nu > 0)$$

递推公式：

$$\left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^m\left[z^\nu \mathrm{I}_\nu(z)\right] = z^{\nu-m}\mathrm{I}_{\nu-m}(z) \quad (m = 0,1,2,\cdots)$$

$$\left(\frac{\mathrm{d}}{z\mathrm{d}z}\right)^m\left[z^{-\nu} \mathrm{I}_\nu(z)\right] = z^{-\nu-m}\mathrm{I}_{\nu+m}(z) \quad (m = 0,1,2,\cdots)$$

$$\mathrm{I}_{\nu-1}(z) + \mathrm{I}_{\nu+1}(z) = 2\mathrm{I}'_\nu(z)$$

$$\mathrm{I}_{\nu-1}(z) - \mathrm{I}_{\nu+1}(z) = \frac{2\nu}{z}\mathrm{I}_\nu(z)$$

近似值：

$$\mathrm{I}_\nu(x) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \quad (\nu > 0,\ x \to 0^+)$$

$$\mathrm{I}_\nu(x) \approx \frac{\mathrm{e}^x}{\sqrt{2\pi x}} \quad (x \to \infty)$$

半奇数阶时：

$$\mathrm{I}_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

$$\mathrm{I}_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

$$\mathrm{I}_{\frac{3}{2}} = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sinh x}{x} - \cosh x\right)$$

$$\mathrm{I}_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cosh x}{x} - \sinh x\right)$$

$$\mathrm{I}_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2}+1\right)\sinh x - \frac{3}{x}\cosh x\right]$$

$$\mathrm{I}_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2}+1\right)\cosh x - \frac{3}{x}\sinh x\right]$$

(5) 第二类修正贝塞尔函数 $\mathrm{K}_\nu(z)$

定义：

$$\mathrm{K}_\nu(z) = \frac{\pi}{2}\frac{\mathrm{I}_{-\nu}(z) - \mathrm{I}_\nu(z)}{\sin\nu\pi}$$

与汉克尔函数的关系：

$$\mathrm{K}_\nu(z) = \frac{\mathrm{i}\pi}{2}\mathrm{e}^{\mathrm{i}\frac{\nu\pi}{2}}\mathrm{H}_\nu^{(1)}\left(z\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) \quad \left(-\pi < \arg z < \frac{\pi}{2}\right)$$

$$\mathrm{K}_\nu(z) = -\frac{\mathrm{i}\pi}{2}\mathrm{e}^{-\mathrm{i}\frac{\nu\pi}{2}}\mathrm{H}_\nu^{(2)}\left(z\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) \quad \left(-\frac{\pi}{2} < \arg z < \pi\right)$$

级数展开：

$$\mathrm{K}_0(z) = -\mathrm{I}_0(z)\left(\ln\frac{z}{2} + \gamma\right) + \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)$$

$$\mathrm{K}_n(z) = (-1)^{n+1}\mathrm{I}_n(z)\left(\ln\frac{z}{2} + \gamma\right) + \frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}$$

$$+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^n}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n}[\psi(k+n+1)+\psi(k+1)]$$

$(n=1,2,3,\cdots)$

积分表达式：

$$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt \quad \left(|\arg z|<\frac{\pi}{2}\right)$$

$$K_\nu(z)=\frac{\sqrt{\pi}\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}dt \quad \left(\mathrm{Re}\,\nu>-\frac{1}{2},\ |\arg z|<\frac{\pi}{2}\right)$$

$$K_n(z)=\frac{(2n)!}{2^n n!z^n}\int_0^\infty\frac{\cos(z\sinh t)}{\cosh^{2n}t}dt \quad (\mathrm{Re}\,z>0,\ n\text{ 为整数})$$

$$K_\nu(x)=\frac{1}{\cos\frac{\nu\pi}{2}}\int_0^\infty\cos(x\sinh t)\cosh\nu t\,dt \quad (x>0,\ -1<\mathrm{Re}\,\nu<1)$$

关系式：

$$K_{-\nu}(z)=K_\nu(z)$$

递推公式：

$$\frac{d}{dz}[z^\nu K_\nu(z)]=-z^\nu K_{\nu-1}(z)$$

$$\frac{d}{dz}[z^{-\nu}K_\nu(z)]=-z^{-\nu}K_{\nu+1}(z)$$

$$K_{\nu-1}(z)+K_{\nu+1}(z)=-2K'_\nu(z)$$

$$K_{\nu-1}(z)-K_{\nu+1}(z)=-\frac{2\nu}{z}K'_\nu(z)$$

近似公式：

$$K_0(x)\approx-\ln x \quad (x\to0^+)$$

$$K_\nu(x)\approx\frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \quad (\nu>0,\ x\to0^+)$$

$$K_\nu(x)\approx\sqrt{\frac{\pi}{2x}}e^{-x} \quad (x\to\infty)$$

半奇数阶时：

$$K_{\frac{1}{2}}(x)=K_{-\frac{1}{2}}(x)=e^{-x}\sqrt{\frac{\pi}{2x}}$$

$$K_{\frac{3}{2}}(x)=K_{-\frac{3}{2}}(x)=e^{-x}\sqrt{\frac{\pi}{2x}}\left(\frac{1}{x}+1\right)$$

$$K_{\frac{5}{2}}(x)=K_{-\frac{5}{2}}(x)=e^{-x}\sqrt{\frac{\pi}{2x}}\left(\frac{2}{x^2}+\frac{2}{x}+1\right)$$

Ⅳ.1.2.11 勒让德(Legendre)函数(球函数)$P_n(x)$和 $Q_n(x)$

勒让德方程

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\nu(\nu+1)y=0$$

的解为勒让德函数. 方程中的 ν 和 x 可以是任何复数,该方程称为 ν 次勒让德方程.

1. ν 为 0 或正整数

当 ν 为 0 或正整数时,勒让德方程成为

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \quad (n=0,1,2,\cdots)$$

它的一个解是多项式,该多项式记作 $P_n(x)$.

(1) 勒让德多项式 $P_n(x)$

$$P_n(x)=\frac{(2n)!}{2^n(n!)^2}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}-\cdots\right]$$

$$=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2n-2k)!x^{n-2k}}{2^nk!(n-k)!(n-2k)!}$$

$P_n(x)$称为**第一类勒让德函数**.

勒让德多项式也可用微商形式的罗德里格斯(Rodrigues)公式表示,即

$$P_n(x)=\frac{1}{2^nn!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

(2) 第一类勒让德函数的积分表述

$$P_n(x)=\frac{1}{2\pi i}\int_C\frac{(t^2-1)^n}{2^n(t-x)^{n+1}}dt$$

这个公式叫做施拉夫利(Schläfli)公式,其中,C 是 t 平面上绕 $t=x$ 点的围道,n 为整数.

勒让德函数也可表示为定积分:

$$P_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(x+\sqrt{x^2-1}\cos\varphi\right)^n d\varphi$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\cos\varphi\right)^n d\varphi$$

这是 $P_n(x)$的拉普拉斯第一积分表示;或者

$$P_n(x)=\frac{1}{\pi}\int_0^{\pi}\frac{d\varphi}{\left(x+\sqrt{x^2-1}\cos\varphi\right)^{n+1}}$$

该式是拉普拉斯第二积分.

(3) $P_n(x)$的递推关系式

$$P_1(x) - xP_0(x) = 0$$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

$$(n = 1,2,3,\cdots)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x) \quad (n = 1,2,3,\cdots)$$

(4) 前几个勒让德多项式 $P_n(x)$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

(5) 勒让德多项式的正交性

$$\int_{-1}^{1} P_n(x)P_k(x)\mathrm{d}x = 0 \quad (k \neq n)$$

$$\int_{-1}^{1} [P_n(x)]^2\mathrm{d}x = \frac{2}{2n+1}$$

2. ν 为正整数

当 ν 为正整数时,可求得勒让德方程

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

的第二个解

$$Q_n(x) = P_n(x)\int_x^{\infty} \frac{\mathrm{d}x}{(x^2-1)[P_n(x)]^2} \quad (|x| > 1)$$

$Q_n(x)$称为**第二类勒让德函数**. 把 $P_n(x)$代入上式的右方,可得到用级数表示的第二类勒让德函数:

$$Q_n(x) = \frac{2^n(n!)^2}{(2n+1)!}x^{-n-1}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)}x^{-2} + \cdots\right] \quad (|x| > 1)$$

(1) 第二类勒让德函数 $Q_n(x)$

$$Q_n(x) = \frac{2^n(n!)^2}{(2n+1)!}x^{-n-1}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)}x^{-2} + \cdots\right] \quad (|x| > 1)$$

或

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{2n-4k-1}{(2k+1)(n-k)}P_{n-2k-1}(x)$$

$(|x|<1,\ n=1,2,3,\cdots)$

(2) $Q_n(x)$的递推关系式

$$Q_1(x) - xQ_0(x) + 1 = 0$$

$$(n+1)Q_{n+1}(x) - (2n+1)xQ_n(x) + nQ_{n-1}(x) = 0$$

$(n=1,2,3,\cdots)$

$$Q'_{n+1}(x) - Q'_{n-1}(x) = (2n+1)Q_n(x) \quad (n=1,2,3,\cdots)$$

Ⅳ.1.2.12 连带勒让德函数 $P_n^m(x)$和 $Q_n^m(x)$

1. 第一类连带勒让德函数 $P_n^m(x)$

连带勒让德方程

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

的解为连带勒让德函数. 在 $n=0,1,2,\cdots$和 m 为任何整数时,连带勒让德函数为

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}[P_n(x)] \quad (-1\leqslant x\leqslant 1,\ n\geqslant m\geqslant 0)$$

$P_n^m(x)$称为 m 阶 n 次的**第一类连带勒让德函数**. 它满足关系式

$$P_n^0(x) = P_n(x) \quad [P_n(x)\text{是勒让德函数}]$$

$$P_n^{-m}(x) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x)$$

2. 第二类连带勒让德函数 $Q_n^m(x)$

连带勒让德方程的另一个解为

$$Q_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}[Q_n(x)] \quad (-1<x<1)$$

$Q_n^m(x)$称为 m 阶 n 次的**第二类连带勒让德函数**.

3. 递推关系

$$(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)$$

$$(2n+1)(1-x^2)^{\frac{1}{2}}P_n^m(x) = P_{n-1}^{m+1}(x) - P_{n+1}^{m+1}(x)$$

$$(2n+1)(1-x^2)^{\frac{1}{2}}P_n^m(x)$$
$$= (n-m+2)(n-m+1)P_{n+1}^{m-1}(x) - (n+m)(n+m-1)P_{n-1}^{m-1}(x)$$

$$(2n+1)(1-x^2)\frac{dP_n^m(x)}{dx}$$
$$= (n+1)(n+m)P_{n-1}^m(x) - n(n-m+1)P_{n+1}^m(x)$$

$Q_n^m(x)$的递推关系与 $P_n^m(x)$相似.

4. 连带勒让德函数的正交性

$$\int_{-1}^{1} P_n^m(x)P_k^m(x)dx = 0 \quad (k \neq n)$$

$$\int_{-1}^{1} [P_n^m(x)]^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}$$

Ⅳ.1.2.13 埃尔米特(Hermite)多项式 $H_n(x)$

1. 定义

埃尔米特方程

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

的一个解为**埃尔米特多项式**

$$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \quad (n = 0,1,2,\cdots)$$

2. 递推公式

$$H_{2n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x)$$

3. 特殊值

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!!$$

$$H_{2n+1}(0) = 0$$

4. 埃尔米特多项式的正交性

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_k(x)dx = 0 \quad (k \neq n)$$

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n n!\sqrt{\pi}$$

Ⅳ.1.2.14 拉盖尔(Laguerre)多项式 $L_n(x)$和连带拉盖尔多项式 $L_n^m(x)$

1. 定义

拉盖尔方程

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0$$

的多项式解为**拉盖尔多项式**

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k n!x^k}{(k!)^2(n-k)!}$$

2. 递推公式

$$(n+1)L_{n+1}(x)+(x-1-2n)L_n(x)+nL_{n-1}(x)=0$$

$$L_n'(x)=L_{n-1}'(x)-L_{n-1}(x)$$

3. 特殊值

$$L_0(x)=1$$

$$L_1(x)=-x+1$$

$$L_2(x)=x^2-4x+2$$

$$L_3(x)=-x^3+9x^2-18x+6$$

$$L_4(x)=x^4-16x^3+72x^2-96x+24$$

4. 拉盖尔多项式的正交性

$$\int_0^\infty e^{-x}L_n(x)L_k(x)dx=0\quad(k\neq n)$$

$$\int_0^\infty e^{-x}[L_n(x)]^2dx=1$$

5. 连带拉盖尔多项式及其性质

(1) 定义

连带拉盖尔多项式为

$$L_n^m(x)=\sum_{k=0}^{n}\frac{(-1)^k(n+m)!x^k}{(n-k)!(m+k)!k!}\quad(m=0,1,2,\cdots)$$

它满足连带拉盖尔方程

$$x\frac{d^2y}{dx^2}+(m+1-x)\frac{dy}{dx}+ny=0$$

(2) 连带拉盖尔多项式与拉盖尔多项式的关系

$$L_n^0(x)=L_n(x)$$

$$L_n^m(x)=(-1)^m\frac{d^m}{dx^m}[L_{n+m}(x)]\quad(m=1,2,3,\cdots)$$

(3) 连带拉盖尔多项式的正交性

$$\int_0^\infty e^{-x}x^mL_n^m(x)L_k^m(x)dx=\begin{cases}0 & (n\neq k)\\ \dfrac{\Gamma(n+m+1)}{n!} & (n=k)\end{cases}$$

Ⅳ.1.2.15 δ函数

1. 定义

δ函数(Dirac Delta Function)的定义为

$$\delta(x) = \begin{cases} 0 & (x \neq 0) \\ \infty & (x = 0) \end{cases}$$

并且满足

$$\int_{-\infty}^{\infty} \delta(x) \mathrm{d}x = 1$$

2. 极限表示

δ函数可用非奇异函数的极限表示,例如:

$$\delta(x) = \lim_{m \to \infty} \frac{\sin mx}{\pi x} \quad \text{(傅里叶定理)}$$

$$\delta(x) = \lim_{\alpha \to \infty} \sqrt{\frac{\alpha}{\pi}} \mathrm{e}^{-\alpha x^2}$$

$$\delta(x) = \lim_{\alpha \to \infty} \sqrt{\frac{\alpha}{\pi}} \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \mathrm{e}^{-\mathrm{i}\alpha x^2}$$

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mathrm{e}^{-\frac{|x|}{\varepsilon}}$$

$$\delta(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2 + \varepsilon^2}$$

$$\lim_{\varepsilon \to 0} \frac{1}{x \pm \mathrm{i}\varepsilon} = (\mathrm{P.V.}) \frac{1}{x} \mp \mathrm{i}\pi\delta(x) = \frac{1}{x \pm \mathrm{i}0}$$

等等.

3. 微商表示

δ函数也可用阶梯函数的微商来表示,设

$$\mathrm{H}(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x > 0) \end{cases}$$

$$\ln(x \pm \mathrm{i}0) = \ln|x| \pm \mathrm{i}\pi \mathrm{H}(-x)$$

则

$$\delta(x) = \mathrm{H}'(x)$$

因此有

$$\frac{\mathrm{d}\ln(x \pm \mathrm{i}0)}{\mathrm{d}x} = \frac{1}{x} \mp \mathrm{i}\pi\,\delta(x)$$

或

$$\delta(x) = \pm \frac{1}{\mathrm{i}\pi x} \mp \frac{1}{\mathrm{i}\pi} \frac{\mathrm{d}\ln(x \pm \mathrm{i}0)}{\mathrm{d}x}$$

4. δ 函数的特性和积分表达式

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x + a) + \delta(x - a)]$$

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)\mathrm{d}x = f(a)$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \quad \text{（傅里叶积分变换）}$$

$$\delta(\rho - \rho') = \rho\int_{0}^{\infty} k\mathrm{J}_m(k\rho)\mathrm{J}_m(k\rho')\mathrm{d}k \quad \text{（傅里叶-贝塞尔积分变换）}$$

Ⅳ.2 常用导数表

1. $\frac{\mathrm{d}}{\mathrm{d}x}(a) = 0$

2. $\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$

3. $\frac{\mathrm{d}}{\mathrm{d}x}(au) = a\frac{\mathrm{d}u}{\mathrm{d}x}$

4. $\frac{\mathrm{d}}{\mathrm{d}x}(u + v - w) = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x}$

5. $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

6. $\frac{\mathrm{d}}{\mathrm{d}x}(uvw) = uv\frac{\mathrm{d}w}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + vw\frac{\mathrm{d}u}{\mathrm{d}x}$

7. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

8. $\frac{\mathrm{d}}{\mathrm{d}x}(u^n) = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$

9. $\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x}$

10. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x}$

11. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}}\frac{\mathrm{d}u}{\mathrm{d}x}$

12. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}}\left(nv\frac{\mathrm{d}u}{\mathrm{d}x} - mu\frac{\mathrm{d}v}{\mathrm{d}x}\right)$

13. $\frac{\mathrm{d}}{\mathrm{d}x}(u^n v^m) = u^{n-1}v^{m-1}\left(nv\frac{\mathrm{d}u}{\mathrm{d}x} + mu\frac{\mathrm{d}v}{\mathrm{d}x}\right)$

14. $\frac{\mathrm{d}}{\mathrm{d}x}[f(u)] = \frac{\mathrm{d}f(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$

15. $\frac{\mathrm{d}^2}{\mathrm{d}x^2}[f(u)] = \frac{\mathrm{d}f(u)}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2f(u)}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2$

16. $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \binom{n}{0}v\frac{\mathrm{d}^nu}{\mathrm{d}x^n} + \binom{n}{1}\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n-1}u}{\mathrm{d}x^{n-1}} + \binom{n}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^{n-2}u}{\mathrm{d}x^{n-2}} + \cdots$

$$+\binom{n}{k}\frac{\mathrm{d}^kv}{\mathrm{d}x^k}\frac{\mathrm{d}^{n-k}u}{\mathrm{d}x^{n-k}} + \cdots + \binom{n}{n}u\frac{\mathrm{d}^nv}{\mathrm{d}x^n}$$

$\left[\text{这里},\binom{n}{r} = \frac{n!}{r!(n-r)!},\ \binom{n}{0} = 1\right]$

17. $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}u}}\quad\left(\frac{\mathrm{d}x}{\mathrm{d}u} \neq 0\right)$

18. $\frac{\mathrm{d}}{\mathrm{d}x}(\log_a u) = \frac{1}{u\ln a}\frac{\mathrm{d}u}{\mathrm{d}x}$

19. $\frac{\mathrm{d}}{\mathrm{d}x}(\ln u) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$

20. $\frac{\mathrm{d}}{\mathrm{d}x}(a^u) = a^u\ln a\frac{\mathrm{d}u}{\mathrm{d}x}\quad(a^u = \mathrm{e}^{u\ln a})$

21. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^u) = \mathrm{e}^u\frac{\mathrm{d}u}{\mathrm{d}x}$

22. $\frac{\mathrm{d}}{\mathrm{d}x}(u^v) = vu^{v-1}\frac{\mathrm{d}u}{\mathrm{d}x} + (\ln u)u^v\frac{\mathrm{d}v}{\mathrm{d}x}$

23. $\frac{\mathrm{d}}{\mathrm{d}x}(\sin u) = \cos u\frac{\mathrm{d}u}{\mathrm{d}x}$

24. $\frac{\mathrm{d}}{\mathrm{d}x}(\cos u) = -\sin u\frac{\mathrm{d}u}{\mathrm{d}x}$

25. $\frac{\mathrm{d}}{\mathrm{d}x}(\tan u) = \sec^2 u\frac{\mathrm{d}u}{\mathrm{d}x}$

26. $\frac{\mathrm{d}}{\mathrm{d}x}(\cot u) = -\csc^2 u\frac{\mathrm{d}u}{\mathrm{d}x}$

27. $\frac{\mathrm{d}}{\mathrm{d}x}(\sec u) = \sec u\tan u\frac{\mathrm{d}u}{\mathrm{d}x}$

28. $\frac{\mathrm{d}}{\mathrm{d}x}(\csc u) = -\csc u\cot u\frac{\mathrm{d}u}{\mathrm{d}x}$

29. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{vers}u) = \sin u\frac{\mathrm{d}u}{\mathrm{d}x}$

（这里，$\mathrm{vers}u = 1 - \cos u$，称为角的正矢）

30. $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \left(-\frac{\pi}{2} \leqslant \arcsin u \leqslant \frac{\pi}{2}\right)$

31. $\frac{\mathrm{d}}{\mathrm{d}x}(\arccos u) = -\frac{1}{\sqrt{1-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} \quad (0 \leqslant \arccos u \leqslant \pi)$

32. $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan u) = \frac{1}{1+u^2}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \left(-\frac{\pi}{2} < \arctan u < \frac{\pi}{2}\right)$

33. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arccot}\, u) = -\frac{1}{1+u^2}\frac{\mathrm{d}u}{\mathrm{d}x} \quad (0 < \mathrm{arccot}\, u < \pi)$

34. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcsec}\, u) = \frac{1}{u\sqrt{u^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \left(0 \leqslant \mathrm{arcsec}\, u < \frac{\pi}{2},\ \frac{\pi}{2} < \mathrm{arcsec}\, u \leqslant \pi\right)$

35. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arccsc}\, u) = -\frac{1}{u\sqrt{u^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \left(-\frac{\pi}{2} \leqslant \mathrm{arccsc}\, u < 0,\ 0 < \mathrm{arccsc}\, u \leqslant \frac{\pi}{2}\right)$

36. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcvers}\, u) = \frac{1}{\sqrt{2u-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x} \quad (0 \leqslant \mathrm{arcvers}\, u \leqslant \pi)$

（这里，$\mathrm{arcvers}u$ 称为反正矢函数）

37. $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = \cosh u \frac{\mathrm{d}u}{\mathrm{d}x}$

38. $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh u) = \sinh u \frac{\mathrm{d}u}{\mathrm{d}x}$

39. $\frac{\mathrm{d}}{\mathrm{d}x}(\tanh u) = \mathrm{sech}^2 u \frac{\mathrm{d}u}{\mathrm{d}x}$

40. $\frac{\mathrm{d}}{\mathrm{d}x}(\coth u) = -\mathrm{csch}^2 u \frac{\mathrm{d}u}{\mathrm{d}x}$

41. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, u) = -\mathrm{sech}\, u \tanh u \frac{\mathrm{d}u}{\mathrm{d}x}$

42. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{csch}\, u) = -\mathrm{csch}\, u \coth u \frac{\mathrm{d}u}{\mathrm{d}x}$

43. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arsinh}\, u) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(u + \sqrt{u^2+1}\right)\right] = \frac{1}{\sqrt{u^2+1}}\frac{\mathrm{d}u}{\mathrm{d}x}$

44. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcosh}\, u) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(u + \sqrt{u^2-1}\right)\right] = \frac{1}{\sqrt{u^2-1}}\frac{\mathrm{d}u}{\mathrm{d}x}$

（$u > 1$，$\mathrm{arcosh}\, u > 0$）

45. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{artanh}\, u) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln\frac{1+u}{1-u}\right) = \frac{1}{1-u^2}\frac{\mathrm{d}u}{\mathrm{d}x} \quad (u^2 < 1)$

46. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcoth}\, u) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln\frac{u+1}{u-1}\right) = -\frac{1}{u^2-1}\frac{\mathrm{d}u}{\mathrm{d}x} \quad (u^2 > 1)$

47. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arsech}\, u) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\frac{1+\sqrt{1-u^2}}{u}\right) = -\frac{1}{u\sqrt{1-u^2}}\frac{\mathrm{d}u}{\mathrm{d}x}$

$(0 < u < 1,\ \mathrm{arsech}u > 0)$

48. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcsch}u) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\frac{1+\sqrt{1+u^2}}{u}\right) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{\mathrm{d}u}{\mathrm{d}x}$

49. $\frac{\mathrm{d}}{\mathrm{d}q}\int_p^q f(x)\mathrm{d}x = f(q)$

50. $\frac{\mathrm{d}}{\mathrm{d}p}\int_p^q f(x)\mathrm{d}x = -f(p)$

51. $\frac{\mathrm{d}}{\mathrm{d}a}\int_p^q f(x,a)\mathrm{d}x = \int_p^q \frac{\partial}{\partial a}[f(x,a)]\mathrm{d}x + f(q,a)\frac{\mathrm{d}q}{\mathrm{d}a} - f(p,a)\frac{\mathrm{d}p}{\mathrm{d}a}$

Ⅳ.3 常用级数展开

Ⅳ.3.1 二项式函数

1. $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + y^n$

[这里，二项式系数 $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$，或 $\binom{n}{k} = \frac{n!}{k!(n-k)!}$，以下同]

2. $(1\pm x)^n = 1 \pm \binom{n}{1}x + \binom{n}{2}x^2 \pm \binom{n}{3}x^3 + \cdots + (\pm 1)^m\binom{n}{m}x^m + \cdots$
$+ (\pm 1)^n x^n \quad (n = 1,2,3,\cdots)$

3. $(1\pm x)^{-n} = 1 \mp \binom{n}{1}x + \binom{n+1}{2}x^2 \mp \binom{n+2}{3}x^3 + \cdots$
$+ (\mp 1)^m\binom{n+m-1}{m}x^m + \cdots \quad (x^2 < 1)$

4. $(1\pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \frac{5}{128}x^4 \pm \cdots \quad (x^2 < 1)$

5. $(1\pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \frac{35}{128}x^4 \mp \cdots \quad (x^2 < 1)$

6. $(1\pm x)^{\frac{1}{3}} = 1 \pm \frac{1}{3}x - \frac{1}{9}x^2 \pm \frac{5}{81}x^3 - \frac{10}{243}x^4 \pm \cdots \quad (x^2 < 1)$

7. $(1 \pm x)^{-\frac{1}{3}} = 1 \mp \frac{1}{3}x + \frac{2}{9}x^2 \mp \frac{14}{81}x^3 + \frac{35}{243}x^4 \mp \cdots \quad (x^2 < 1)$

8. $(1 \pm x)^{\frac{3}{2}} = 1 \pm \frac{3}{2}x + \frac{3}{8}x^2 \mp \frac{3}{48}x^3 + \frac{3}{128}x^4 \mp \frac{15}{1280}x^5 + \cdots \quad (x^2 < 1)$

9. $(1 \pm x)^{-\frac{3}{2}} = 1 \mp \frac{3}{2}x + \frac{15}{8}x^2 \mp \frac{105}{48}x^3 + \frac{945}{384}x^4 \mp \cdots \quad (x^2 < 1)$

10. $(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \quad (x^2 < 1)$

11. $(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots \quad (x^2 < 1)$

12. $(1 \mp x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} (\pm x)^k \quad (x^2 < 1)$

$\left[\text{这里}, \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, \alpha \text{ 为任何实数}\right]$

Ⅳ.3.2 指数函数

1. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$

2. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

3. $a^x = e^{x \ln a} = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \cdots + \frac{(x\ln a)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(x\ln a)^n}{n!}$

4. $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

5. $e^{e^x} = e\left(1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \cdots\right)$

6. $e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \cdots$

7. $e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right)$

8. $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \cdots \quad \left(x^2 < \frac{\pi^2}{4}\right)$

9. $e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{5x^4}{4!} + \cdots$

10. $e^{\arctan x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \cdots$

Ⅳ.3.3 对数函数

1. $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \quad (2 \geqslant x > 0)$$

2. $\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right]$

$$= 2\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1} \quad (x > 0)$$

3. $\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-1}{x}\right)^n \quad \left(x \geqslant \frac{1}{2}\right)$

4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad (1 \geqslant x > -1)$

5. $\ln(1-x) = -\left(x + \frac{x^2}{x} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) = -\sum_{n=1}^{\infty}\frac{x^n}{n} \quad (-1 \leqslant x < 1)$

6. $\ln(n+1) = \ln(n-1) + 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right)$

7. $\ln(a+x) = \ln a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \cdots\right.$

$$\left. + \frac{1}{2n+1}\left(\frac{x}{2a+x}\right)^{2n+1} + \cdots\right] \quad (a > 0, -a < x)$$

8. $\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right)$

$$= 2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1} \quad (|x| < 1)$$

9. $\ln\frac{x+1}{x-1} = 2\left[\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots + \frac{1}{(2n+1)x^{2n+1}} + \cdots\right]$

$$= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)x^{2n+1}} \quad (|x| > 1)$$

10. $\ln(\sin x) = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots + (-1)^n\frac{2^{2n-1}B_{2n}}{n(2n)!}x^{2n} + \cdots$

$$= \ln x + \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}B_{2n}}{n(2n)!}x^{2n} \quad (0 < x^2 < \pi^2)$$

（这里，B_{2n} 为第 $2n$ 次伯努利数，以下同）

11. $\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots + (-1)^n\frac{2^{2n-1}(2^{2n-1}-1)B_{2n}}{n(2n)!}x^{2n} + \cdots$

$$= \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n-1}(2^{2n-1}-1)B_{2n}}{n(2n)!}x^{2n} \quad \left(x^2 < \frac{\pi^2}{4}\right)$$

12. $$\ln(\tan x) = \ln x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots + (-1)^{n-1}\frac{2^{2n}(2^{2n-1}-1)B_{2n}}{n(2n)!}x^{2n} + \cdots$$

$$= \ln x + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}(2^{2n-1}-1)B_{2n}}{n(2n)!}x^{2n} \quad \left(0 < x^2 < \frac{\pi^2}{4}\right)$$

Ⅳ.3.4 三角函数

1. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$$

2. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

3. $$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} \quad \left(x^2 < \frac{\pi^2}{4}\right)$$

（这里，B_{2n} 为第 $2n$ 次伯努利数，以下同）

4. $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots + (-1)^n\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1} + \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1} \quad (0 < x^2 < \pi^2)$$

5. $$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots + (-1)^n\frac{E_{2n}}{(2n)!}x^{2n} + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n} \quad \left(x^2 < \frac{\pi^2}{4}\right)$$

（这里，E_{2n} 为第 $2n$ 次欧拉数）

6. $$\csc x = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots + (-1)^{n+1}\frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}x^{2n-1} \quad (0 < x^2 < \pi^2)$$

Ⅳ.3.5 反三角函数

1. $$\arcsin x = x + \frac{1}{2\cdot 3}x^3 + \frac{1.3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots$$

$$+ \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1} \quad \left(x^2 < 1, -\frac{\pi}{2} < \arcsin x < \frac{\pi}{2}\right)$$

2. $$\arccos x = \frac{\pi}{2} - \left[x + \frac{1}{2 \cdot 3}x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \cdots + \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1} + \cdots\right]$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1} \quad (x^2 < 1, 0 < \arccos x < \pi)$$

3. $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} \quad (x^2 < 1)$$

4. $$\operatorname{arccot} x = \frac{\pi}{2} - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots\right]$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} \quad (x^2 < 1)$$

Ⅳ.3.6 双曲函数

1. $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

2. $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

3. $$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1} \quad \left(|x| < \frac{\pi}{2}\right)$$

4. $$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots + \frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1} - \cdots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} x^{2n-1} \quad (0 < |x| < \pi)$$

5. $$\operatorname{sech} x = 1 - \frac{1}{2!}x^2 + \frac{5}{4!}x^4 - \frac{61}{6!}x^6 + \frac{1385}{8!}x^8 - \cdots + \frac{E_{2n}}{(2n)!}x^{2n} - \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n} \quad \left(|x| < \frac{\pi}{2}\right)$$

6. $$\operatorname{csch} x = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots - \frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots$$

$$= \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)B_{2n}}{(2n)!} x^{2n-1} \quad (0 < |x| < \pi)$$

7. $\sinh ax = \frac{2}{\pi}\sinh\pi a\left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - \cdots\right) \quad (|x|<\pi)$

8. $\cosh ax = \frac{2a}{\pi}\sinh\pi a\left(\frac{1}{2a^2} - \frac{\cos x}{a^2+1^2} + \frac{\cos 2x}{a^2+2^2} - \frac{\cos 3x}{a^2+3^2} + \cdots\right) \quad (|x|<\pi)$

9. $\sinh nu = \sinh u\left[(2\cosh u)^{n-1} - \frac{n-2}{1!}(2\cosh u)^{n-3} + \frac{(n-3)(n-4)}{2!}(2\cosh u)^{n-5}\right.$
$\left. - \frac{(n-4)(n-5)(n-6)}{3!}(2\cosh u)^{n-7} + \cdots\right]$

10. $\cosh nu = \frac{1}{2}\left[(2\cosh u)^{n} - \frac{n}{1!}(2\cosh u)^{n-2} + \frac{n(n-3)}{2}(2\cosh u)^{n-4}\right.$
$\left. - \frac{n(n-4)(n-5)}{3!}(2\cosh u)^{n-6} + \cdots\right]$

Ⅳ.3.7 反双曲函数

1. $\operatorname{arsinh} x = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots$
$+ (-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1} + \cdots$
$= \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1} \quad (|x|<1)$

2. $\operatorname{arcosh} x = \ln(2x) - \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{6x^6} - \cdots - \frac{(2n)!}{2^{2n}(n!)^2}\frac{1}{2nx^{2n}} - \cdots$
$= \ln(2x) - \sum_{n=1}^{\infty}\frac{(2n-1)!}{2^{2n}(n!)^2}x^{-2n} \quad (|x|>1)$

3. $\operatorname{artanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$
$= \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1} \quad (|x|<1)$

4. $\operatorname{arcoth} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots + \frac{1}{(2n+1)x^{2n+1}} + \cdots$
$= \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{-(2n+1)} \quad (|x|>1)$

5. $\operatorname{arsech} x = \ln\frac{2}{x} - \frac{1}{2}\frac{x^2}{2} - \frac{1\cdot 3}{2\cdot 4}\frac{x^4}{4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^6}{6} - \cdots - \frac{(2n)!}{2^{2n}(n!)^2}\frac{x^{2n}}{2n} - \cdots$
$= \ln\frac{2}{x} - \sum_{n=1}^{\infty}\frac{(2n-1)!}{2^{2n}(n!)^2}x^{2n} \quad (0<x<1)$

6. $\operatorname{arcsch} x = \frac{1}{x} - \frac{1}{2}\frac{1}{3x^3} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{5x^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7x^7} + \cdots$

$$+(-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{(2n+1)x^{2n+1}} + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{-(2n+1)} \quad (|x|>1)$$

Ⅳ.4 自然科学基本常数

Ⅳ.4.1 数学常数

Ⅳ.4.1.1 常数 π(圆周率)

圆周率 π 被定义为圆的周长与直径之比,即 C(圆周长)$=2\pi r$,其中 $2r$ 为直径.利用圆周率 π 和半径 r 可得到下列有用公式:

$$A(\text{圆面积}) = \pi r^2$$

$$V(\text{球体积}) = \frac{4}{3}\pi r^3$$

$$SA(\text{球面积}) = 4\pi r^2$$

π 的近似值为

π≈ 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679 (前 100 位)

Ⅳ.4.1.2 常数 e(自然对数之底)

自然对数之底 e 由下式定义:

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}$$

e 的近似值为

e≈ 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 95749 66967 62772 40766 30353 54759 45713 82178 52516 64274 (前 100 位)

函数 e^x 的定义为

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

e 和 π 的关系式为

$$e^{\pi i} = -1$$

$$e^{2\pi i} = 1$$

e^{π} 和 π^e 的近似值为

$e^{\pi} \approx 23.14069\ 26327\ 79269\ 00572\ 90863\ 67948\ 54738\ 02661\ 06242\ 60021$
（前 50 位）

$\pi^e \approx 22.45915\ 77183\ 61045\ 47342\ 71522\ 04543\ 73502\ 75893\ 15133\ 99669$
（前 50 位）

Ⅳ.4.1.3 欧拉(Euler)常数 γ

欧拉常数 γ 被定义为

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right)$$

γ 的近似值为

$\gamma \approx 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93992$
（前 50 位）

Ⅳ.4.1.4 黄金分割比例常数 φ

黄金分割比例常数 φ 被定义为方程 $\frac{1}{\varphi} = \frac{\varphi}{1-\varphi}$ 的正根，也就是

$$\varphi = \frac{\sqrt{5}-1}{2}$$

φ 的近似值为

$\varphi \approx 0.61803\ 39887\ 49894\ 84820\ 45868\ 34365\ 63811\ 77203\ 09179\ 80576$
（前 50 位）

Ⅳ.4.1.5 卡塔兰(Catalan)常数 G

卡塔兰常数 G 定义为

$$G = \frac{1}{2}\int_0^1 \mathrm{K}\,\mathrm{d}k = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2}$$

其中，K 为完全椭圆积分：

$$\mathrm{K} \equiv \mathrm{K}(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

G 的近似值为

$$G = 0.915965594\cdots$$

Ⅳ.4.1.6 伯努利(Bernoulli)多项式 $B_n(x)$和伯努利数 B_n

伯努利多项式 $B_n(x)$是用母函数

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

定义的. 伯努利数 B_n 是伯努利多项式中的系数 $B_n(x)$在 $x=0$ 处的值,即 $B_n = B_n(0)$,因此伯努利数的母函数为

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{\mathrm{e}^t - 1}$$

前 13 位伯努利数 B_n 分别为

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	$\cdots$
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	$\cdots$

伯努利数 B_n 中,除 B_1 外的所有奇数项皆为 0. 因此,公式中常常只使用偶数项的伯努利数 B_{2n},当 $n=0,1,2,\cdots$时,伯努利数分别是 $B_0, B_2, B_4, \cdots$.

Ⅳ.4.1.7 欧拉(Euler)多项式 $E_n(x)$和欧拉数 E_n

欧拉多项式 $E_n(x)$是用母函数

$$\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

定义的. 欧拉数 E_n 是欧拉多项式中的系数 $E_n(x)$在 $x = \frac{1}{2}$ 处的值,即 $E_n = 2^n E_n\left(\frac{1}{2}\right)$,因此欧拉数 E_n 的母函数为

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2\mathrm{e}^t}{\mathrm{e}^{2t} + 1} = \frac{1}{\cosh t}$$

前 11 位欧拉数 E_n 为

E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	$\cdots$
1	0	-1	0	5	0	-61	0	1385	0	-50521	$\cdots$

欧拉数 E_n 的所有奇数项皆为 0,只有偶数项才有数值. 因此,公式中经常只使用偶数项的欧拉数 E_{2n},当 $n=0,1,2,\cdots$时,欧拉数分别是 $E_0, E_2, E_4, \cdots$.

Ⅳ.4.2 物理学常数

物理量	符号	数值	单位	备注
真空中的光速	c	2.99792458×10^{8}	m/s	
普朗克常数	h	$6.62606896(33)\times10^{-34}$	J·s	
约化普朗克常数	$\hbar=\frac{h}{2\pi}$	$1.054571628(53)\times10^{-34}$	J·s	
电子电荷	e	$1.602176487(40)\times10^{-19}$	C	
		$4.80320427(12)\times10^{-10}$	esu	
电子的荷质比	$\frac{e}{m_e}$	1.75882015×10^{11}	C/kg	
电子的静止质量	m_e	$9.10938215(45)\times10^{-31}$	kg	
		$0.510998910(13)$	MeV/c^2	
质子的静止质量	m_p	$1.672621637(83)\times10^{-27}$	kg	
		$938.272013(23)$	MeV/c^2	
中子的静止质量	m_n	1.67494101×10^{-27}	kg	
		939.5731	MeV/c^2	
电子的经典半径	r_e	$2.8179402894(58)\times10^{-15}$	m	
质子的经典半径	r_p	1.534698×10^{-18}	m	
自由空间的介电常数	$\varepsilon_0=\frac{1}{\mu_0 c^2}$	$8.854187817\times10^{-12}$	F/m	
自由空间的磁导率	μ_0	$1.2566370614\times10^{-6}$	H/m	
精细结构常数	$\alpha=\frac{e^2}{4\pi\varepsilon_0\hbar c}$	$\frac{1}{137.035999679(94)}$		
阿伏伽德罗常数	N_A	$6.02214179(30)\times10^{23}$	mol^{-1}	
玻耳兹曼常数	k	$1.3806504(24)\times10^{-23}$	J/K	
标准重力加速度	g_n	9.80665	m/s^2	
里德伯常数	R_∞	1.097373177×10^{7}	m^{-1}	
声速(大气中)	V_A	340.5	m/s	15 ℃，10%湿度

选自 AMSLER C, et al. Particle Physics Booklet[M]. LBNL&CERN,2008.

Ⅳ.4.3 化学常数(元素周期表)

元素周期表

图 例

原子序数 → 22

Ti ← 元素符号

元素名称 → 钛

47.87 ← 原子量

Group	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
周期																			
1	1 H 氢 1.008																		2 He 氦 4.003
2	3 Li 锂 6.941	4 Be 铍 9.012												5 B 硼 10.81	6 C 碳 12.01	7 N 氮 14.01	8 O 氧 16.00	9 F 氟 19.00	10 Ne 氖 20.18
3	11 Na 钠 22.99	12 Mg 镁 24.31												13 Al 铝 26.98	14 Si 硅 28.09	15 P 磷 30.97	16 S 硫 32.07	17 Cl 氯 35.45	18 Ar 氩 39.95
4	19 K 钾 39.10	20 Ca 钙 40.08		21 Sc 钪 44.96	22 Ti 钛 47.87	23 V 钒 50.94	24 Cr 铬 52.00	25 Mn 锰 54.94	26 Fe 铁 55.85	27 Co 钴 58.93	28 Ni 镍 58.69	29 Cu 铜 63.55	30 Zn 锌 65.41	31 Ga 镓 69.72	32 Ge 锗 72.61	33 As 砷 74.92	34 Se 硒 78.96	35 Br 溴 79.90	36 Kr 氪 83.80
5	37 Rb 铷 85.47	38 Sr 锶 87.62		39 Y 钇 88.91	40 Zr 锆 91.22	41 Nb 铌 92.91	42 Mo 钼 95.94	43 Tc 锝 98.91	44 Ru 钌 101.1	45 Rh 铑 102.9	46 Pd 钯 106.4	47 Ag 银 107.9	48 Cd 镉 112.4	49 In 铟 114.8	50 Sn 锡 118.7	51 Sb 锑 121.8	52 Te 碲 127.6	53 I 碘 126.9	54 Xe 氙 131.3
6	55 Cs 铯 132.9	56 Ba 钡 137.3	56-70 镧系 *	71 Lu 镥 175.0	72 Hf 铪 178.5	73 Ta 钽 180.9	74 W 钨 183.8	75 Re 铼 186.2	76 Os 锇 190.2	77 Ir 铱 192.2	78 Pt 铂 195.1	79 Au 金 197.0	80 Hg 汞 200.6	81 Tl 铊 204.4	82 Pb 铅 207.2	83 Bi 铋 209.0	84 Po 钋 (210)	85 At 砹 (210)	86 Rn 氡 (222)
7	87 Fr 钫 (223)	88 Ra 镭 (226)	89-102 锕系 **	103 Lr 铹* (260)	104 Rf (260)	105 Db (262)	106 Sg (263)	107 Bh (262)	108 Hs (265)	109 Mt (266)	110 Uun (269)	111 Uuu (272)	112 Uub (277)	113 Uut	114 Uuq (289)	115 Uup	116 Uuh (289)	117 Uus	118 Uuo (293)

镧系	57 La 镧 138.9	58 Ce 铈 140.1	59 Pr 镨 140.9	60 Nd 钕 144.2	61 Pm 钷 144.9	62 Sm 钐 150.4	63 Eu 铕 152.0	64 Gd 钆 157.3	65 Tb 铽 158.9	66 Dy 镝 162.5	67 Ho 钬 164.9	68 Er 铒 167.3	69 Tm 铥 168.9	70 Yb 镱 173.0
锕系	89 Ac 锕 (227)	90 Th 钍 232.0	91 Pa 镤 231.0	92 U 铀 238.0	93 Np 镎 (237)	94 Pu 钚 (239)	95 Am 镅* (243)	96 Cm 锔* (247)	97 Bk 锫* (247)	98 Cf 锎* (252)	99 Es 锿* (252)	100 Fm 镄* (257)	101 Md 钔* (256)	102 No 锘* (259)

Ⅳ.4.4 天文学常数

名　称	符　号	数　值	单　位
光速	c	299792458	m/s
牛顿重力常数	G_N	$6.6743(7)\times10^{-11}$	$m^3/(kg\cdot s^2)$
天文单位	AU	149597870700(3)	m
回归年	yr	31556925.2	s
恒星年		31558149.8	s
平均恒星日		23h56m04.09053s	
普朗克质量	$\sqrt{\frac{\hbar c}{G_N}}$	$2.17644(11)\times10^{-8}$ $1.22089(6)\times10^{19}$	kg GeV/c^2
普朗克长度	$\sqrt{\frac{\hbar G_N}{c^3}}$	$1.61624(8)\times10^{-35}$	m
哈勃长度	$\frac{c}{H_0}$	$\sim1.27\times10^{26}$	m
秒差距	pc	$3.0856775807\times10^{16}$	m
光年	ly	$0.946053\cdots\times10^{16}$	m
太阳的史瓦西半径	$\frac{2G_N M_\odot}{c^2}$	2.9532500770(2)	km
太阳质量	$M_\odot$	$1.9884(2)\times10^{30}$	kg
太阳赤道半径	$R_\odot$	$6.9551(3)\times10^{8}$	m
太阳光度	$L_\odot$	$3.8427(14)\times10^{26}$	W
太阳到银河系中心的距离	R_0	8.0(5) 24.68542064×10^{16}	kpc km
太阳绕银河系中心的运动速度	$v_\odot$	220(20)	km/s
哈勃常数	H_0	$100h$	km/(s·pc)
归一化哈勃常数	h	0.73(3)	
宇宙的临界密度	$\rho_c=\frac{3H_0{}^2}{8\pi G_N}$	$1.87835(19)\times10^{-26}\times h^2$	kg/m^3
宇宙年龄	t_0	$13.73(15)\times10^{9}$	yr(回归年)
宇宙背景辐射(CBR)温度	T_0	2.725 ± 0.001	K
太阳相对于 CBR 的速度		369(2)	km/s

选自 AMSLER C, et al. Particle Physics Booklet[M]. LBNL&CERN,2008.

几个非国际单位制的常用单位的说明：

(1) 电子伏特(eV),能量单位,为 1 个单电荷粒子经过 1 V 电位差所获得的能量,1 eV = 1.6×10^{-19} J.

(2) 秒差距 parsec(pc),为 1 天文单位的距离所张的角为 1 角秒时的距离.

Ⅳ.4.5 地学常数

特　性	整个地球	地　核
地球赤道半径 a	6378.137 km	3488 km
地球地极半径 c	6356.752 km	3479 km
地球扁率 $e=\frac{a-c}{a}$	$\frac{1}{298.2572}$	$\frac{1}{390}$
地球平均半径 $\sqrt[3]{a^2c}$	6371.00 km	3485 km
地球质量	5.976×10^{24} kg	1.88×10^{24} kg
地球平均密度	5518 kg/m^3	10720 kg/m^3
地球自旋角速度	7.2921152×10^{-5} rad/s	
地球表面积	5.101×10^{14} m^2	1.52×10^{14} m^2
地球体积	1.083×10^{21} m^3	0.176×10^{21} m^3
地球子午线的四分之一长度	10002.002 km	5640 km
地球陆地面积	1.49×10^{14} m^2	
地球陆地平均高度	840 m	
地球陆地最高高度	8840 m	
地球海洋面积	3.61×10^{14} m^2	
地球海洋体积	1.37×10^{18} m^3	
地球海水质量	1.42×10^{21} kg	
地球海洋平均深度	3800 m	
地球海洋最深深度	10550 m	
地球大气质量	5.27×10^{18} kg	
地球标准重力加速度	9.80665 m/s^2	

选自 KAYE G W C, LABY T H. Tables of Physical and Chemical Constants(物理和化学常数表)[M]. 北京:世界图书出版公司,1999.

第一宇宙速度(first cosmic velocity)

$$v_1=7.9\times10^3\ \text{m/s}$$

第二宇宙速度(second cosmic velocity)

$$v_2 = 11.2 \times 10^3 \text{ m/s}$$

第三宇宙速度(third cosmic velocity)

$$v_3 = 16.7 \times 10^3 \text{ m/s}$$

Ⅳ.5 国际单位制(SI)

Ⅳ.5.1 国际单位制(SI)中十进制倍数和词头表示法

倍数	词头		符号	常用名称
	英文	中文		
$10^{10^{100}}$				googolplex
10^{100}				googol
10^{24}	yotta	尧[它]	Y	heptillion
10^{21}	zetta	泽[它]	Z	hexillion
10^{18}	exa	艾[可萨]	E	quintillion
10^{15}	peta	拍[它]	P	quadrillion
10^{12}	tera	太[拉]	T	trillion
10^{9}	giga	吉[咖]	G	billion
10^{6}	mega	兆	M	million
10^{3}	kilo	千	k	thousand
10^{2}	hecto	百	h	hundred
10^{1}	deca	十	da	ten
10^{-1}	deci	分	d	tenth
10^{-2}	centi	厘	c	hundredth
10^{-3}	milli	毫	m	thousandth
10^{-6}	micro	微	μ	millionth
10^{-9}	nano	纳[诺]	n	billionth
10^{-12}	pico	皮[可]	p	trillionth
10^{-15}	femto	飞[母托]	f	quadrillionth
10^{-18}	atto	阿[托]	a	quintillionth
10^{-21}	zepto	仄[普托]	z	hexillionth
10^{-24}	yocto	么[科托]	y	heptillionth

Ⅳ.5.2 国际单位制(SI)的基本单位

量的名称	单位名称	单位符号
长度	米	m
质量	千克(公斤)	kg
时间	秒	s
电流	安[培]	A
热力学温度	开[尔文]	K
物质的量	摩[尔]	mol
发光强度	坎[德拉]	cd

Ⅳ.5.3 国际单位制(SI)中具有专门名称的导出单位

量的名称	单位名称	单位符号	用其他国际制单位表示的关系式	用国际制基本单位表示的关系式
[平面]角	弧度	rad	1	
立体角	球面度	sr	1	
频率	赫兹	Hz		s^{-1}
力	牛顿	N		$m \cdot kg \cdot s^{-2}$
压力,压强,应力	帕斯卡	Pa	N/m^2	$m^{-1} \cdot kg \cdot s^{-2}$
能[量],功,热量	焦耳	J	$N \cdot m$	$m^2 \cdot kg \cdot s^{-2}$
功率,辐[射能]通量	瓦特	W	J/s	$m^2 \cdot kg \cdot s^{-3}$
电量,电荷	库仑	C		$s \cdot A$
电压,电位,电动势	伏特	V	W/A	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-1}$
电容	法拉	F	C/V	$m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2$
电阻	欧姆	Ω	V/A	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-2}$
电导	西门子	S	A/V	$m^{-2} \cdot kg^{-1} \cdot s^3 \cdot A^2$
磁通[量]	韦伯	Wb	$V \cdot s$	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-1}$
磁感应强度	特斯拉	T	Wb/m^2	$kg \cdot s^{-2} \cdot A^{-1}$
电感	亨利	H	Wb/A	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-2}$
摄氏温度	摄氏度	℃		
光通量	流明	lm		$cd \cdot sr$
光照度	勒克斯	lx	lm/m^2	$m^{-2} \cdot cd \cdot sr$
[放射性]活度	贝可勒尔	Bq		s^{-1}
吸收剂量	戈瑞	Gy	J/kg	$m^2 \cdot s^{-2}$
剂量当量	希沃特	Sv	J/kg	$m^2 \cdot s^{-2}$

符号索引

1. 特殊函数的符号

B_n, B_{2k}　伯努利数

$B_n(x)$　伯努利多项式

$\mathrm{B}(x, y)$　贝塔函数

$\mathrm{C}(x)$　菲涅耳余弦积分函数

$\mathrm{Ci}(x), \mathrm{ci}(x)$　余弦积分

E_n, E_{2k}　欧拉数

$E_n(x)$　欧拉多项式

$\mathrm{E}(k)=\mathrm{E}$　第二类完全椭圆积分

$\mathrm{Ei}(z)$　指数积分

$\mathrm{erf}(x)=\Phi(x)$　误差函数(概率积分函数)

$\mathrm{erfc}(x)$　补余误差函数

$\mathrm{F}(k, \varphi)$　勒让德第一类椭圆积分

$\mathrm{E}(k, \varphi)$　勒让德第二类椭圆积分

$\Pi(h, k, \varphi)$　勒让德第三类椭圆积分

G　卡塔兰常数

γ　欧拉常数

$\Gamma(z)$　伽马函数

$\mathrm{H}_\nu^{(1)}(z), \mathrm{H}_\nu^{(2)}(z)$　第一类和第二类汉克尔函数,第三类贝塞尔函数

$\mathrm{H}_n(x)$　埃尔米特多项式

$\mathrm{I}_\nu(z)$　第一类修正贝塞尔函数

$\mathrm{J}_\nu(z)$　第一类贝塞尔函数

$\mathrm{K}(k)=\mathrm{K}$, $\mathrm{K}(k')=\mathrm{K}'$　第一类完全椭圆积分

$\mathrm{K}_\nu(z)$　第二类修正贝塞尔函数

$\mathrm{L}_n(x)$　拉盖尔多项式

$\mathrm{L}_n^m(x)$　连带拉盖尔多项式

$\mathrm{li}(x)$　对数积分

$\mathrm{N}_\nu(z)$　诺伊曼函数(第二类贝塞尔函数)

$P_\nu(z), P_n(x)$ 勒让德函数和勒让德多项式

$P_\nu^m(z), P_n^m(x)$ 连带勒让德函数和连带勒让德多项式

$\Phi(x)$ 概率积分函数

$\psi(x)$ 普赛函数

$Q_\nu(z), Q_n(x)$ 第二类勒让德函数和勒让德多项式

$Q_\nu^m(z), Q_n^m(x)$ 第二类连带勒让德函数和连带勒让德多项式

$S(x)$ 菲涅耳正弦积分函数

$Si(x), si(x)$ 正弦积分

$Z_\nu(z)$ 贝塞尔函数

2. 本书中几个常用的数学符号

$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ n 的阶乘，n 为大于零的正整数. $0! = 1! = 1$

$(2n+1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)$ $2n+1$ 的双阶乘. $1!! = 1, (-1)!! = 1$

$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)$ $2n$ 的双阶乘. $0!! = 1, 2!! = 2$

$i = \sqrt{-1}$ 虚数单位. $i^2 = -1, i^3 = -i, i^4 = 1, i^{4n+1} = i$

$z = a + ib$ z 为复数，a 和 b 分别为复数的实部和虚部，记作 $a = \mathrm{Re}z, b = \mathrm{Im}z$

$$\arg z = \begin{cases} \arctan\dfrac{b}{a} & (a > 0) \\ \pi + \arctan\dfrac{b}{a} & (a < 0,\ b > 0) \\ -\pi + \arctan\dfrac{b}{a} & (a < 0,\ b < 0) \end{cases}$$ 复数 z 的辐角

$C_n^k = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ 二项式系数

$C_\alpha^k = \binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ 推广的二项式系数（α 为任意实数）. $C_\alpha^0 = 1$

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